On quantum dynamics of the self-acting electron

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Abstract

The paper deals with the quantum theory of the self-organizing electrically charged matter in the nonrelativistic approximation. Nonrelativistic limit of the fundamental dynamical equation, describing the behaviour of the self-acting electron, is investigated and energy characteristics of the system of interacting fields are obtained. The Ehrenfest theorems and the quantum Newton equations for the self-acting electron in arbitrary external field are derived and with their aid the non-stationary states of the particles are investigated. The wave functions of a free electron, corresponding to the lowest energy stationary states, as well as the wave functions, describing the stationary states of the electron in homogeneous electric and magnetic fields, are calculated. It is shown that the stationary state of the electron in the homogeneous and constant external field is a superposition of solitons that differ from each other in geometric shape and linear dimensions.

Introduction

This paper is a development and continuation of studies in quantum mechanics and quantum electrodynamics (QED) taking into account the Coulomb self-acting of electrically charged particles [1-10]. Basic to approach being developed is the idea of the electron as a self-organized elementary excitation of the charged matter field, whose geometric shape and size are determined in a self-consistent way from a nonlinear dynamical equation. Apparently quantum mechanics of self-acting electron can be considered as the simplest example of a theory of self-organization in physical systems (see [11-12]). The notions of physical mechanisms of self-organization (in particular, of formation of particles as solitons) as well as the methods of solving the fundamental dynamical equation, developed therein, are of universal nature and can be useful not only for studying systems with electromagnetic interaction.

In the formulation of quantum mechanics being developed the electron wave function has direct physical meaning: it describes the spatial distribution of the particle electric charge. The assumption that direct physical interpretation of quantum mechanics is possible was put forward for the first time by Schrödinger [13, 14]. The construction of a consistent quantum model of self-acting electron [8] and the explanation of Balmer's spectrum of hydrogen atom [4, 6, 10] within the theory of two self-acting bodies can serve as the basis for the Schrödinger interpretation. Note that during the last few decades the Schrödinger hypothesis attracted attention of many researches in connection with a new approach to calculation of radiative corrections not using the notion of vacuum fluctuations and also in connection with an attempt to reformulate QED using only the self-energy picture [18-21]. As is emphasized in the paper by Barut and van Huele [19], the correct quantum equation of motion of electron has to take into account the self-energy of the particle.

In Section 1 of this paper the nonrelativistic limit of the fundamental dynamical equation obtained and investigated in [7, 10] is considered. The Lagrangian and Hamiltonian functions of the nonrelativistic self-acting electron are constructed in Section 2. Section 3 deals with the energy

characteristics of the self-acting charged field in the nonrelativistic approximation. In Section 4 the generalization of the Ehrenfest theorem for the self-acting electron taking into account the particle spin is given. It is shown that the average self-acting force, acting on the particle, is a quantity of the order of α^2 (α is the fine structure constant). In Section 5 the wave functions describing the non-stationary states of the electron in an arbitrary homogeneous external field are obtained. The calculation of the wave functions of the stationary state both for the free self-acting particle and for the particle in an external field is carried out in Sections 6-8.

According to [9] the wave function of a stationary state of the electron in homogeneous external field is a superposition of an infinite number of spherical harmonics, whose amplitudes describe solitons. Ina sufficiently weak external field the main contribution to the mentioned superposition is given only by the first two harmonics. In the Appendix the transformation properties of the wave function of self-acting electron under Galilean transformation are considered.

1. The equation of motion of the electron field in the nonrelativistic approximation

According to [7, 10] the quantum relativistic of motion of the electric field, taking account of its self-action, is given by

$$\left(i\hat{\partial} - e\beta\hat{A} - M\right)\begin{pmatrix}\Psi\\ \tilde{\Psi}\end{pmatrix} = 0 \qquad (1)$$

where β is the self-action constant, e and M are the charge and the mass of the electron; $A^\mu = A^\mu_{||} + A^\mu_\perp$, $A^\mu_{||}$ and $A^\mu_\perp$ are the potential and the vortex components of the 4-potential

$$\begin{aligned} A^\mu_{||}(x) &= \int d^4x'\delta\left((x-x')^2\right)j^\mu_{||}(x'),\\ A^\mu_\perp(x) &= \frac{1}{2}\left(\mathcal{A}^\mu_\perp(x) + \tilde{\mathcal{A}}^\mu_\perp(x)\right), \qquad (2)\\ \mathcal{A}^\mu_\perp(x) &= -\frac{1}{4\pi}\int d^4x'\delta\left((x-x')^2\right)\partial_\nu{}'\mathcal{F}^{\mu\nu}_\perp(x'),\\ \tilde{\mathcal{A}}^\mu_\perp(x) &= -\frac{1}{4\pi}\int d^4x'\delta\left((x-x')^2\right)\partial_\nu{}'\tilde{\mathcal{F}}^{\mu\nu}_\perp(x') \end{aligned}$$

where $\mathcal{F}^{\mu\nu}_\perp$ and $\tilde{\mathcal{F}}^{\mu\nu}_\perp$ are the vortex components of the 4-tensors $\mathcal{F}^{\mu\nu}$ and $\tilde{\mathcal{F}}^{\mu\nu}$ of electromagnetic field; $j^\mu_{||}$ is the potential component of the 4-current density j^μ,

$$j^\mu(x) = e\left(\overline{\tilde{\Psi}}(x)\gamma^\mu\Psi(x) + \overline{\Psi}(x)\gamma^\mu\tilde{\Psi}(x)\right)$$

Further, we shall assume that the components of the wave functions Ψ and $\tilde{\Psi}$ are connected to each other by the equality (see, for example, [9]),

$$\tilde{\Psi} = a\Psi, \quad a = const \qquad (3)$$

Therefore we can restrict our consideration to the equation of motion for one component only (for example, for Ψ). It is convenient to denote

$$e\beta A^\mu = e\beta\left(A^0, \vec{A}\right) = \left(\mathcal{A}^0, \vec{\mathcal{A}}\right) \qquad (4)$$

and to rewrite the equation of motion (1) as $\left(\vec{\alpha} = \gamma^0, \vec{\gamma}\right)$

$$\left(i\frac{\partial}{\partial t} + \vec{\alpha}\left(i\frac{\partial}{\partial t} + \vec{\mathcal{A}}\right) - \mathcal{A}^0 - \gamma^0 M\right)\Psi = 0 \qquad (5)$$

To go to nonrelativistic limit let's put

$$\Psi = e^{-iMt}\begin{pmatrix}\phi\\ \chi\end{pmatrix} \qquad (6)$$

The substitution of (6) into Eq.(5) leads to the following set of equations for the spinors ϕ and χ:

$$\left(i\frac{\partial}{\partial t} - \mathscr{A}^0\right)\phi + \vec{\sigma}\left(i\frac{\partial}{\partial t} + \vec{\mathscr{A}}\right)\chi = 0 \qquad (7)$$
$$\vec{\sigma}\left(i\frac{\partial}{\partial t} + \vec{\mathscr{A}}\right)\phi + \left(i\frac{\partial}{\partial t} - \mathscr{A}^0 + 2M\right)\chi = 0$$

Taking into account that

$$\frac{1}{2M}\left(i\frac{\partial}{\partial t} - \mathscr{A}^0\right) \approx \alpha^2$$

(α is the fine structure constant) and neglecting the value of the order of α^2, from the second of Eqs. (7) we can derive:

$$\chi = -\frac{1}{2M}\vec{\sigma}\left(i\frac{\partial}{\partial t} + \vec{\mathscr{A}}\right)\phi \qquad (8)$$

Substituting the expression (8) into the first of Eqs. (7) leads to

$$\left\{i\frac{\partial}{\partial t} - \mathscr{A}^0 - \frac{1}{2M}\left[\vec{\sigma}\left(i\frac{\partial}{\partial t} + \vec{\mathscr{A}}\right)\right]^2\right\}\phi = 0$$

Taking into account the notation (4) and the operator identity

$$\left[\vec{\sigma}\left(i\frac{\partial}{\partial \vec{r}} + \vec{\mathscr{A}}\right)\right]^2 = \left(-i\frac{\partial}{\partial \vec{r}} - \vec{\mathscr{A}}\right)^2 - \left(\vec{\sigma}\left[\frac{\partial}{\partial \vec{r}}\vec{\mathscr{A}}\right]\right),$$

it is easy to recast the last equation in the form (here we reconstruct both of the wave function components):

$$i\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \tilde{\phi}\end{pmatrix} = \left\{\frac{1}{2M}\left(-i\frac{\partial}{\partial t} - e\beta\vec{A}\right)^2 - \frac{e\beta}{2M}\vec{\sigma}\vec{H} + e\beta A^0\right\}\begin{pmatrix}\phi\\ \tilde{\phi}\end{pmatrix} \qquad (9)$$

where $\vec{H} = rot\vec{A}$, Eq. (9) is just the desired equation of motion in the nonrelativistic approximation. Note that in the nonrelativistic approximation it is convenient to use such a gauge of the 4-potential, in which its potential and vortex components are given by (see [8])

$$A^{\mu}_{\parallel}(x) = (\varphi(x), 0), \qquad A^{\mu}_{\perp}(x) = (0, \vec{A}(x)),$$
$$\varphi(x) = \int d\vec{r}\,'\left|\vec{r} - \vec{r}\,'\right|^{-1}\rho(\vec{r}\,', t),$$
$$\vec{A}(x) = \frac{1}{2}\left(\vec{\mathscr{A}}_{\perp}(x) + \tilde{\vec{\mathscr{A}}}_{\perp}(x)\right), \qquad (10)$$
$$\vec{\mathscr{A}}_{\perp}(x) = \frac{1}{4\pi}\int d\vec{r}\,'\left|\vec{r} - \vec{r}\,'\right|^{-1}\left(\vec{\nabla}' \times \vec{B}(x')\right)_{t'=t},$$
$$\tilde{\vec{\mathscr{A}}}_{\perp}(x) = \frac{1}{4\pi}\int d\vec{r}\,'\left|\vec{r} - \vec{r}\,'\right|^{-1}\left(\vec{\nabla}' \times \tilde{\vec{B}}(x')\right)_{t'=t},$$
$$\rho(x) = e\left(\tilde{\phi}^{+}(x)\phi(x) + \phi^{+}(x)\tilde{\phi}(x)\right)$$

The last term in curly brackets in Eq. (9) has the meaning of the potential energy of the self-acting electron and can be written in the form:

$$e\beta A^0(\vec{r}, t) = e^2\beta N\int d\vec{r}\,'\left|\vec{r} - \vec{r}\,'\right|^{-1}\phi^{+}(\vec{r}\,', t)\phi(\vec{r}\,', t) \equiv U(\vec{r}, t) \qquad (11)$$

where the constraint $\tilde{\phi} = a\phi$, $a = const$ (see (3)) and the notation $a + a^{*} = N \quad (= \pm 1)$ have been used.

In the absence of the interaction of the electron with the vortex electromagnetic field Eq. (9) takes the form

$$i\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \tilde{\phi}\end{pmatrix}=\left(-\frac{1}{2M}\vec{\nabla}^2+U(\vec{r},t)\right)\begin{pmatrix}\phi\\ \tilde{\phi}\end{pmatrix} \qquad (12)$$

In the case when the self-acting electron interacts not only with the vortex electromagnetic field but also with an external field described by the 4-potential $A_{ext}=(\varphi_{ext},\vec{A}_{ext})$, it is natural to describe the electron motion by Eq. (9), in which the following substitution is made

$$\beta A^0\to\beta A^0+\varphi_{ext},\ \beta\vec{A}\to\beta\vec{A}+\vec{A}_{ext},$$
$$\beta\vec{H}\to\beta\vec{H}+\vec{H}_{ext},\ \vec{H}_{ext}=[\vec{\nabla}\times\vec{A}_{ext}]$$

We proceed from the fact that the self-acting electron field cannot influence the external field and therefore it is natural to include the self-action constant β in the definition of the external field potentials.

2. The Lagrangian and Hamiltonian functions

One can obtain the equations of motion (9) using the action principle $\delta S=0$, $S=\int dtL$, with the Lagrangian function

$$L=L_{01}+L_{02}+L_{\mathrm{int},1}+L_{\mathrm{int},2}, \qquad (13)$$

$$L_{01}=\int d\vec{r}\sum_{\alpha=1,2}\left\{\tilde{\phi}^*_\alpha\frac{i}{2}\frac{\vec{\partial}}{\partial t}\phi_\alpha+\phi^*_\alpha\frac{i}{2}\frac{\vec{\partial}}{\partial t}\tilde{\phi}_\alpha-\frac{1}{2M}\left[\left(\vec{\pi}_0\tilde{\phi}_\alpha\right)^*\left(\vec{\pi}_\alpha\phi_\alpha\right)+\left(\vec{\pi}_0\phi_\alpha\right)\left(\vec{\pi}_0\tilde{\phi}_\alpha\right)\right]\right\};$$

$$L_{02}=\frac{\beta}{8\pi}\int d\vec{r}\left[\dot{\vec{\tilde{\mathscr{A}}}}_\perp\dot{\vec{\mathscr{A}}}_\perp-\sum_{\alpha=1,2,3}\left(\nabla_\alpha\vec{\tilde{\mathscr{A}}}_\perp\right)\left(\nabla_\alpha\vec{\mathscr{A}}_\perp\right)\right];$$

$$L_{\mathrm{int},1}=-\frac{\beta}{2}\int d\vec{r}_1\int d\vec{r}_2\left|\vec{r}_1-\vec{r}_2\right|^{-1}\rho(\vec{r}_1,t)\rho(\vec{r}_2,t); \qquad (14)$$

$$L_{\mathrm{int},2}=-\frac{1}{2M}\sum_{\alpha=1,2}\int d\vec{r}\left[\left(\vec{\pi}\tilde{\phi}_\alpha\right)^*\left(\vec{\pi}\ \phi_\alpha\right)+\left(\vec{\pi}\phi_\alpha\right)^*\left(\vec{\pi}\tilde{\phi}_\alpha\right)-\left(\vec{\pi}_0\tilde{\phi}_\alpha\right)^*\left(\vec{\pi}_0\phi_\alpha\right)-\left(\vec{\pi}_0\phi_\alpha\right)^*\left(\vec{\pi}_0\tilde{\phi}_\alpha\right)\right]+$$
$$+\frac{e\beta}{2M}\sum_{\alpha,\beta=1,2}\int d\vec{r}\left(\tilde{\phi}^*_\alpha\vec{\sigma}_{\alpha\beta}\vec{H}\phi_\beta+\phi^*_\alpha\vec{\sigma}_{\alpha\beta}\vec{H}\tilde{\phi}_\beta\right),$$

where $\vec{\pi}_0=-i\vec{\nabla}$; $\vec{\pi}=-i\vec{\nabla}-e\beta\vec{A}$; $\phi_\alpha\ (\alpha=1,2)$ are the components of the spinor ϕ. Varying the function of action S with respect to the dynamical variables $\dot{\vec{\mathscr{A}}}_\perp$ and $\dot{\vec{\tilde{\mathscr{A}}}}_\perp$ of the vortex electromagnetic field leads to the Maxwell equations

$$\left(\partial_t^2-\vec{\nabla}^2\right)\vec{\mathscr{A}}_\perp=\left(\partial_t^2-\vec{\nabla}^2\right)\vec{\tilde{\mathscr{A}}}_\perp=4\pi\vec{j}_\perp \qquad (15)$$

which can be transformed to the 4-dimensional form

$$\partial_\nu\mathscr{F}^{\mu\nu}_\perp=\partial_\nu\tilde{\mathscr{F}}^{\mu\nu}_\perp=-4\pi j^\mu_\perp$$

Here and in the following the usual rule of summation over two repeating indices one of which is upper and the other is lower, is meant; $\vec{j}_\perp$ is the vortex component of the current density vector $\vec{j}$,

$$\vec{j}=-\frac{ie}{2M}\left(\tilde{\phi}^+\vec{\nabla}\phi+\phi^+\vec{\nabla}\tilde{\phi}\right)-\frac{e^2\beta}{M}\vec{A}\left(\tilde{\phi}^+\phi+\phi^+\tilde{\phi}\right)+\vec{j}_{spin};$$
$$\vec{j}_{spin}=\frac{e}{2M}\left[\vec{\nabla}\times\left(\tilde{\phi}^+\vec{\sigma}\phi+\phi^+\vec{\sigma}\tilde{\phi}\right)\right] \qquad (16)$$

Note that the spin component of the current density vector $\vec{j}_{spin}$ is the vortex one: $\vec{\nabla}\vec{j}_{spin} = 0$.
It is easy to generalize the formulae (13), (14) and (16) to the case of the set of n self-acting particles [3]. The Lagrangian function of such a set of particles can be written in the form (13), where L_{02} and $L_{\text{int},1}$ are defined by the previous formulae (14) and L_{01} and $L_{\text{int},2}$ are expressed by

$$L_{01} = \sum_{k=1}^{n} \int d\vec{r} \left\{ \tilde{\phi}_k^{+} \frac{i}{2} \frac{\vec{\partial}}{\partial t} \phi_k + \phi_k^{+} \frac{i}{2} \frac{\vec{\partial}}{\partial t} \tilde{\phi}_k - \frac{1}{2m_k} \left[\left(\vec{\pi}_0 \tilde{\phi}_k\right)^{+} \left(\vec{\pi}_0 \phi_k\right) + \left(\vec{\pi}_0 \phi_k\right)^{+} \left(\vec{\pi}_0 \tilde{\phi}_k\right) \right] \right\}; \qquad (17)$$

$$L_{\text{int},2} = -\sum_{k=1}^{n} \frac{1}{2m_k} \int d\vec{r} \left[\left(\vec{\pi}_k \tilde{\phi}_k\right)^{+} \left(\vec{\pi}_k \phi_k\right) + \left(\vec{\pi}_k \phi_k\right)^{+} \left(\vec{\pi}_k \tilde{\phi}_k\right) - \left(\vec{\pi}_0 \tilde{\phi}_k\right)^{+} \left(\vec{\pi}_0 \phi_k\right) - \left(\vec{\pi}_0 \phi_k\right)^{+} \left(\vec{\pi}_0 \tilde{\phi}_k\right) \right] +$$

$$+ \sum_{k=1}^{n} \frac{e_k \beta}{2m_k} \int d\vec{r} \left(\tilde{\phi}_k^{+} \vec{\sigma} \vec{H} \phi_k + \phi_k^{+} \vec{\sigma} \vec{H} \tilde{\phi}_k \right)$$

Here $\left(\phi_k, \tilde{\phi}_k\right)$ are the wave function components of the particle with mass m_k and charge e_k, $\vec{\pi}_k = -i\vec{\nabla} - e_k \beta \vec{A}$,

$$\rho(\vec{r},t) = \sum_{k=1}^{n} e_k \left(\tilde{\phi}_k^{+}(\vec{r},t) \phi_k(\vec{r},t) + c.c. \right) \qquad (18)$$

is the electric charge density. The action principle for the system is question leads to the following equations of motion $(k = 1,2...,n)$:

$$i \frac{\partial}{\partial t} \begin{pmatrix} \phi_k \\ \tilde{\phi}_k \end{pmatrix} = \left\{ \frac{1}{2m_k} \left(-i \frac{\partial}{\partial \vec{r}} - e_k \beta \vec{A} \right)^2 - \frac{e_k \beta}{2m_k} \vec{\sigma} \vec{H} + U_k \right\} \begin{pmatrix} \phi_k \\ \tilde{\phi}_k \end{pmatrix} \qquad (19)$$

where $U_k = U_k(\vec{r},t)$ is the potential energy of the particle k,

$$U_k(\vec{r},t) = e_k \beta \int d\vec{r}\,' \left| \vec{r} - \vec{r}\,' \right|^{-1} \rho(\vec{r}\,',t) \qquad (20)$$

The total potential energy of the system, including the potential energy of self-action of each particle and the energy of interaction between the particles, is given by formulae

$$W = -L_{\text{int},1} = \frac{1}{2} \sum_{k=1}^{n} \int d\vec{r}\, U_k(\vec{r},t) \left(\tilde{\phi}_k^{+}(\vec{r},t) \phi_k(\vec{r},t) + c.c. \right) =$$

$$= \frac{\beta}{2} \int d\vec{r}_1 \int d\vec{r}_2 \left| \vec{r}_1 - \vec{r}_2 \right|^{-1} \rho(\vec{r}_1,t) \rho(\vec{r}_2,t) \qquad (21)$$

The vector of the electric current density is of the form:

$$\vec{j} = \sum_{k=1}^{n} \frac{1}{2m_k} \left\{ -ie_k \left(\tilde{\phi}_k^{+} \vec{\vec{\nabla}} \phi_k + \phi_k^{+} \vec{\vec{\nabla}} \tilde{\phi}_k \right) - \right. \qquad (22)$$

$$\left. - 2e_k^2 \beta \vec{A} \left(\tilde{\phi}_k^{+} \phi_k + c.c. \right) + e_k \left[\vec{\nabla} \times \left(\tilde{\phi}_k^{+} \vec{\sigma} \phi_k + \phi_k^{+} \vec{\sigma} \tilde{\phi}_k \right) \right] \right\}$$

As the components of the electron wave function ϕ_α and $\tilde{\phi}_\alpha$ are complex, one can consider ϕ_α, $\tilde{\phi}_\alpha$, ϕ_α^{*} and $\tilde{\phi}_\alpha^{*}$ as the independent generalized coordinates of the electron field. The corresponding generalized momenta are given by

$$\Pi_\alpha(x) = \frac{\delta L}{\delta \dot{\phi}_\alpha(x)} = \frac{i}{2} \tilde{\phi}_\alpha^{*}(x); \qquad \tilde{\Pi}_\alpha(x) = \frac{\delta L}{\delta \dot{\tilde{\phi}}_\alpha(x)} = \frac{i}{2} \phi_\alpha^{*}(x);$$

$$\Pi_\alpha^{*}(x) = \frac{\delta L}{\delta \dot{\phi}_\alpha^{*}(x)} = -\frac{i}{2} \tilde{\phi}_\alpha(x); \qquad \tilde{\Pi}_\alpha^{*}(x) = \frac{\delta L}{\delta \dot{\tilde{\phi}}_\alpha^{*}(x)} = -\frac{i}{2} \phi_\alpha(x); \qquad (23)$$

To generalized coordinates $\vec{\mathscr{A}}_\perp$ and $\vec{\tilde{\mathscr{A}}}_\perp$ of the vortex electromagnetic field there correspond the following generalized momenta

$$\vec{\Pi}_\perp(x)=\frac{\delta L}{\delta\dot{\vec{\mathscr{A}}}_\perp(x)}=\frac{\beta}{8\pi}\dot{\vec{\mathscr{A}}}_\perp(x);\qquad \dot{\vec{\Pi}}_\perp(x)=\frac{\delta L}{\delta\ddot{\vec{\mathscr{A}}}_\perp(x)}=\frac{\beta}{8\pi}\dot{\vec{\mathscr{A}}}_\perp(x) \tag{24}$$

The Hamiltonian function of the system of interacting fields is defined by

$$H=\sum_{\alpha=1,2}\int d\vec{r}\left(\Pi_\alpha(x)\dot{\phi}_\alpha(x)+\tilde{\Pi}_\alpha(x)\dot{\tilde{\phi}}_\alpha(x)+\dot{\phi}^*_\alpha(x)\Pi^*_\alpha(x)+\dot{\tilde{\phi}}^*_\alpha(x)\tilde{\Pi}^*_\alpha(x)\right)+$$
$$+\int d\vec{r}\left(\vec{\Pi}_\perp(x)\dot{\vec{\mathscr{A}}}_\perp(x)+\dot{\vec{\mathscr{A}}}_\perp(x)\dot{\vec{\Pi}}_\perp(x)\right)-L=H_{01}+H_{02}+H_{\mathrm{int},1}+H_{\mathrm{int},2} \tag{25}$$

The Hamiltonian function can be expressed in terms of generalized coordinates ϕ_α and $\tilde{\phi}_\alpha$ and generalized momenta Π_a and $\tilde{\Pi}_a$ of electron field:

$$H_{01}=-\frac{i}{M}\sum_{\alpha=1,2}\int d\vec{r}\left[(\vec{\pi}^*_0\Pi_\alpha)(\vec{\pi}_0\phi_\alpha)+(\vec{\pi}^*_0\tilde{\Pi}_\alpha)(\vec{\pi}_0\tilde{\phi}_\alpha)\right];$$
$$H_{02}=\frac{\beta}{8\pi}\int d\vec{r}\left[\left(\frac{8\pi}{\beta}\right)^2\dot{\vec{\Pi}}_\perp\vec{\Pi}_\perp+\sum_{\alpha=1,2,3}(\nabla_\alpha\dot{\vec{\mathscr{A}}}_\perp)(\nabla_\alpha\vec{\mathscr{A}}_\perp)\right];$$
$$H_{\mathrm{int},1}=\frac{\beta}{2}\int d\vec{r}_1\int d\vec{r}_2\,|\vec{r}_1-\vec{r}_2|^{-1}\rho(\vec{r}_1,t)\rho(\vec{r}_2,t); \tag{26}$$
$$H_{\mathrm{int},2}=-\frac{i}{2}\sum_{\alpha=1,2}\int d\vec{r}\left[(\vec{\pi}^*\Pi_\alpha)(\vec{\pi}\phi_\alpha)+(\vec{\pi}^*\tilde{\Pi}_\alpha)(\vec{\pi}\tilde{\phi}_\alpha)-(\vec{\pi}^*_0\Pi_\alpha)(\vec{\pi}_0\phi_\alpha)-(\vec{\pi}^*_0\tilde{\Pi}_\alpha)(\vec{\pi}_0\tilde{\phi}_\alpha)\right]+$$
$$+\frac{ie\beta}{M}\sum_{\alpha,\beta=1,2}\int d\vec{r}\left(\Pi_\alpha\vec{\sigma}_{\alpha\beta}\vec{H}\phi_\beta+\tilde{\Pi}_\alpha\vec{\sigma}_{\alpha\beta}\vec{H}\tilde{\phi}_\beta\right);$$
$$\rho(x)=-2ie\sum_{\alpha=1,2}\left(\Pi_\alpha\phi_\alpha+\tilde{\Pi}_\alpha\tilde{\phi}_\alpha\right)$$

It is easy to check up that the Hamiltonian equations

$$\dot{\phi}_\alpha(x)=\frac{1}{2}\frac{\delta H}{\delta\Pi_\alpha(x)};\qquad \dot{\tilde{\phi}}_\alpha(x)=\frac{1}{2}\frac{\delta H}{\delta\tilde{\Pi}_\alpha(x)};$$
$$\dot{\Pi}_\alpha(x)=-\frac{1}{2}\frac{\delta H}{\delta\phi_\alpha(x)};\qquad \dot{\tilde{\Pi}}_\alpha(x)-\frac{1}{2}\frac{\delta H}{\delta\tilde{\phi}_\alpha(x)}$$

coincide with the equations (9) and the equations

$$\dot{\vec{\Pi}}_\perp=-\frac{\delta H}{\delta\vec{\mathscr{A}}_\perp(x)},\qquad \dot{\vec{\Pi}}_\perp=-\frac{\delta H}{\delta\ddot{\vec{\mathscr{A}}}_\perp(x)}$$

coincide with the Maxwell equations (15).

Using formulae (23) and (24) for the generalized momenta, the components (26) of Hamiltonian function be represented in the form

$$H_{01}+H_{\mathrm{int},2}=\frac{1}{2M}\int d\vec{r}\left[(\vec{\pi}^*\tilde{\phi}^+)(\vec{\pi}\phi)+(\vec{\pi}^*\phi^+)(\vec{\pi}\tilde{\phi})\right]-\frac{e\beta}{2M}\int d\vec{r}\,(\tilde{\phi}^+\vec{\sigma}\vec{H}\phi+\phi^+\vec{\sigma}\vec{H}\tilde{\phi});$$
$$H_{02}=\frac{\beta}{8\pi}\int d\vec{r}\left[\dot{\vec{\mathscr{A}}}_\perp\dot{\vec{\mathscr{A}}}_\perp+\sum_{\alpha=1,2,3}(\nabla_\alpha\dot{\vec{\mathscr{A}}}_\perp)(\nabla_\alpha\vec{\mathscr{A}}_\perp)\right]; \tag{27}$$
$$H_{\mathrm{int},1}=\frac{\beta}{2}\int d\vec{r}_1\int d\vec{r}_2\,|\vec{r}_1-\vec{r}_2|^{-1}\rho(\vec{r}_1,t)\rho(\vec{r}_2,t),$$
$$\rho(\vec{r},t)=e(\tilde{\phi}^+\phi+\phi^+\tilde{\phi})$$

With the Maxwell equations (15) we can find

$$\frac{d}{dt}H_{02}=\frac{\beta}{2}\int d\vec{r}\;\vec{j}_\perp\left(\dot{\vec{\mathscr{A}}}_\perp+\dot{\vec{\mathscr{A}}}_\perp\right)=-\frac{\beta}{2}\int d\vec{r}\;\vec{j}_\perp\left(\vec{E}_\perp+\dot{\vec{E}}_\perp\right) \tag{28}$$

where the equalities

$$\vec{E}_{\perp} = -\dot{\vec{\mathscr{A}}}_{\perp}, \qquad \tilde{\vec{E}}_{\perp} = -\dot{\tilde{\vec{\mathscr{A}}}}_{\perp} \tag{29}$$

are taken into account.

With the aid of the equations of motion (9) one can obtain

$$\frac{d}{dt}\left(H_{01} + H_{\mathrm{int},2}\right) = \frac{\beta}{2}\int d\vec{r}\ \vec{j}_{\perp}\left(\vec{E}_{\perp} + \tilde{\vec{E}}_{\perp}\right) + \tag{30}$$

$$+ \frac{ie\beta}{2M}\int d\vec{r}\left\{\tilde{\phi}^{+}\left[\vec{\nabla}^{2}\overset{\mapsto}{A}^{0} + 2\left(\vec{\nabla}\overset{\mapsto}{A}^{0}\right)\vec{\nabla} - 2ie\beta\,(\vec{A}\vec{\nabla})\overset{\mapsto}{A}^{0}\right]\phi + \left(\tilde{\phi} \Leftrightarrow \phi\right)\right\}$$

To transform the last addend in the right-hand side of the expression (30) one uses the following auxiliary formulae which can easily be obtained by the integration by parts:

$$\int d\vec{r}\,\tilde{\phi}^{+}\left(\vec{\nabla}^{2}\overset{\mapsto}{A}^{0}\right)\phi = -\int d\vec{r}\left(\vec{\nabla}\overset{\mapsto}{A}^{0}\right)\left[\tilde{\phi}^{+}\vec{\nabla}\phi + \left(\vec{\nabla}\tilde{\phi}^{+}\right)\phi\right];$$

$$\int d\vec{r}\,\tilde{\phi}^{+}\left(\vec{\nabla}^{2}\overset{\mapsto}{A}^{0} + 2\left(\vec{\nabla}\overset{\mapsto}{A}^{0}\right)\vec{\nabla}\right)\phi = -\int d\vec{r}\ A^{0}\vec{\nabla}\left(\tilde{\phi}^{+}\tilde{\vec{\nabla}}\phi\right);$$

$$\int d\vec{r}\,\tilde{\phi}^{+}\,2ie\beta(\vec{A}\vec{\nabla})\overset{\mapsto}{A}^{0}\,\phi = -2ie\beta\int d\vec{r}\ A^{0}\vec{\nabla}\left(\vec{A}\tilde{\phi}^{+}\phi\right)$$

While deriving the last equality it has been taken into account that $\vec{A}$ is the vortex vector (i. e. $\vec{\nabla}\vec{A} = 0$).

Considering the auxiliary formulae and the continuity equation $\partial\rho/\partial t = -\vec{\nabla}\vec{j}$, the relationship (30) can be written in the form

$$\frac{d}{dt}\left(H_{01} + H_{\mathrm{int},2}\right) = \frac{\beta}{2}\int d\vec{r}\ \vec{j}_{\perp}\left(\vec{E}_{\perp} + \tilde{\vec{E}}_{\perp}\right) - \beta\int d\vec{r}\,\frac{\partial\rho}{\partial t}\varphi \tag{31}$$

where $\varphi = \varphi(x)$ is the scalar potential (10). Further we obtain

$$\frac{d}{dt}H_{\mathrm{int},1} = \beta\int d\vec{r}\,\frac{\partial\rho}{\partial t}\varphi \tag{32}$$

From Eqs. (28), (31) and (32) follows the law of energy conservation for the total system of interacting fields:

$$H = H_{01} + H_{02} + H_{\mathrm{int},1} + H_{\mathrm{int},2} = const \tag{33}$$

3. The energy-momentum tensor of the interacting fields in the nonrelativistic approximation

Let us define the Lagrangian function density $\mathscr{L} = \mathscr{L}(x)$ and its components by formulae

$$L = \int d\vec{r}\ \mathscr{L}, \qquad L_{0n} = \int d\vec{r}\ \mathscr{L}_{0n},$$

$$L_{\mathrm{int}\,n} = \int d\vec{r}\ \mathscr{L}_{\mathrm{int}\,n} \qquad (n = 1,2)$$

The peculiarity of the function $\mathscr{L}(x)$ in the problem under consideration is that its component $\mathscr{L}_{\mathrm{int},1}$ depends on ϕ_{α} and $\tilde{\phi}_{\alpha}$ both locally and no locally and there is in addition the explicit dependence on position vector $\vec{r}$. To take into account this peculiarity let us separately calculate the quantities $\partial_n\mathscr{L}'$ and $\partial_n\mathscr{L}_{\mathrm{int},1}$, where the function $\mathscr{L}' = \mathscr{L}_{01} + \mathscr{L}_{02} + \mathscr{L}_{\mathrm{int},2}$ depends locally on dynamical variables and their first derivatives with respect to space-time coordinates. One has

$$\partial_\nu \mathscr{L}_{\text{int},1} = -\frac{\beta}{2}\partial_\nu(\rho\varphi) \qquad (34)$$

$$\partial_\nu \mathscr{L}' = \sum_{\alpha=1,2}\left\{\left(\frac{\partial \mathscr{L}'}{\partial\phi_\alpha}\partial_\nu\phi_\alpha + \frac{\partial \mathscr{L}'}{\partial\tilde{\phi}_\alpha}\partial_\alpha\tilde{\phi}_\alpha + c.c.\right)+\right.$$

$$\left.+\sum_\mu\left[\frac{\partial \mathscr{L}'}{\partial(\partial_\mu\phi_\alpha)}\partial_\nu\partial_\mu\phi_\alpha + \frac{\partial \mathscr{L}'}{\partial(\partial_\mu\tilde{\phi}_\alpha)}\partial_\nu\partial_\mu\tilde{\phi}_\alpha + c.c.\right]\right\}+ \qquad (35)$$

$$+\frac{\partial \mathscr{L}'}{\partial\vec{\mathscr{A}}_\perp}\partial_\nu\vec{\mathscr{A}}_\perp + \frac{\partial \mathscr{L}'}{\partial\tilde{\vec{\mathscr{A}}}_\perp}\partial_\nu\tilde{\vec{\mathscr{A}}}_\perp + \sum_\mu\left[\frac{\partial \mathscr{L}'}{\partial(\partial_\mu\vec{\mathscr{A}}_\perp)}\partial_\nu\partial_\mu\vec{\mathscr{A}}_\perp + \frac{\partial \mathscr{L}'}{\partial(\partial_\mu\tilde{\vec{\mathscr{A}}}_\perp)}\partial_\nu\partial_\mu\tilde{\vec{\mathscr{A}}}_\perp\right]$$

The equations of motion can be written using the functional derivatives of the action integral $S = \int dt\, L$:

$$\frac{\delta S}{\delta\phi_\alpha(x)} - \sum_\mu\partial_\mu\left(\frac{\delta S}{\delta(\partial_\mu\phi_\alpha(x))}\right) = 0,$$

$$\frac{\delta S}{\delta\mathscr{A}_\perp^\alpha} - \sum_\mu\partial_\mu\left(\frac{\delta S}{\delta(\partial_\mu\mathscr{A}_\perp^\alpha)}\right) = 0 \qquad (36)$$

etc.

It can be easily verified that the relationship between the functional derivatives of the action function S and the partial derivatives of $\mathscr{L}'$ with respect to generalized coordinates of electron field is given by equalities

$$\frac{\delta S}{\delta\phi_\alpha(x)} = \frac{\partial \mathscr{L}'}{\partial\,\phi_\alpha(x)} - e\beta\,\tilde{\phi}_\alpha^*(x)\varphi(x), \qquad (37)$$

$$\frac{\delta S}{\delta\tilde{\phi}_\alpha(x)} = \frac{\partial \mathscr{L}'}{\partial\,\tilde{\phi}_\alpha(x)} - e\beta\,\phi_\alpha^*(x)\varphi(x),$$

the functional derivatives of S coinciding with the partial derivatives of $\mathscr{L}'$ with respect to all the other dynamical variables, for example

$$\frac{\delta S}{\delta(\partial_\nu\phi_\alpha(x))} = \frac{\partial \mathscr{L}'}{\partial(\partial_\nu\phi_\alpha(x))}; \qquad \frac{\delta S}{\delta\mathscr{A}_\perp^\alpha(x)} = \frac{\partial \mathscr{L}'}{\partial\,\mathscr{A}_\perp^\alpha(x)} \qquad (38)$$

etc.

Taking into account Eqs. (34)-(38) one gets the formula

$$\partial_\nu\mathscr{L} = \sum_\mu\partial_\mu\left\{\sum_\alpha\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_\alpha)}\partial_\nu\phi_\alpha + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\tilde{\phi}_\alpha)}\partial_\nu\tilde{\phi}_\alpha + c.c.\right)+\right.$$

$$\left.+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\vec{\mathscr{A}}_\perp)}\partial_\nu\vec{\mathscr{A}}_\perp + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\tilde{\vec{\mathscr{A}}}_\perp)}\partial_\nu\tilde{\vec{\mathscr{A}}}_\perp\right\} - \frac{\beta}{2}\rho\,\vec{\partial}_\nu\varphi$$

which is conveniently represented in such a form

$$\partial_\nu t^{\mu\nu} = \frac{\beta}{2}\rho\vec{\partial}^\mu\varphi \qquad (39)$$

where

$$t^{\mu\nu} = \sum_{\alpha=1,2}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_\alpha)}\partial^\mu\phi_\alpha + \frac{\partial\mathscr{L}}{\partial(\partial_\nu\tilde{\phi}_\alpha)}\partial^\mu\tilde{\phi}_\alpha + c.c.\right)+$$

$$\frac{\partial\mathscr{L}}{\partial(\partial_\nu\vec{\mathscr{A}}_\perp)}\partial^\mu\vec{\mathscr{A}}_\perp + \frac{\partial\mathscr{L}}{\partial(\partial_\nu\tilde{\vec{\mathscr{A}}}_\perp)}\partial^\mu\tilde{\vec{\mathscr{A}}}_\perp - g^{\mu\nu}\mathscr{L} \qquad (40)$$

Let us introduce the quantity

$$\theta_{||}^{\mu\nu} = -\frac{1}{4\pi}\left(F_{||}^{\mu\alpha} F_{||\alpha}^{\ \nu} - \frac{1}{4} g^{\mu\nu} F_{||}^{\alpha\beta} F_{||\alpha\beta} \right) \qquad (41)$$

which is essentially the potential component of the energy-momentum tensor of the electromagnetic field (the quantity (41) can be obtained from the energy-momentum tensor of electromagnetic field if one drops in it the field $\vec{H}$ and the vortex component of the field $\vec{E}$ (see [8]). Taking into account the expression

$$F_{||}^{\mu\nu} = -\delta_{\mu 0} E_{||\nu} + \delta_{\nu 0} E_{||\mu},$$

which is valid on condition $E_{||0} = 0$, it is easy to obtain the following representation:

$$\theta_{||}^{\mu\nu} = -\frac{1}{4\pi}\left(E_{||\mu} E_{||\nu} - \frac{1}{2}\delta_{\mu\nu}\vec{E}_{||}^{\,2} \right) \qquad (42)$$

The equality

$$\partial_\nu \theta_{||}^{\mu\nu} = -F_{||}^{\mu\alpha} j_{||\alpha}, \qquad (43)$$

is fulfilled.

When calculating the right-hand side of the equality (39) we use the formula

$$\rho \vec{\partial}^\mu \varphi = j_{||\alpha} \vec{\partial}^\mu A_{||}^\alpha, \qquad (44)$$

which obviously takes place for the following gauge of the potential

$$A_{||}^\alpha = (\varphi, 0) \qquad (45)$$

with the identity

$$\frac{1}{2} j_{||\alpha} \vec{\partial}^\mu A_{||}^\alpha = F_{||}^{\mu\alpha} j_{||\alpha} + \partial_\alpha \left(A_{||}^\mu j_{||}^\alpha - \frac{1}{2} g^{\mu\alpha} j_{||\beta} A_{||}^\beta \right)$$

which can be easily verified and the equality (43) one obtains

$$\frac{1}{2} j_{||\alpha} \vec{\partial}^\mu A_{||}^\alpha = \partial_\nu \left(-\theta_{||}^{\mu\nu} + A_{||}^\mu j_{||}^\nu - \frac{1}{2} j_{||\alpha} A_{||}^\alpha \right)$$

When choosing the gauge of the potential in the form (45) the last equality is written in the following way

$$\frac{1}{2}\rho \vec{\partial}^\mu \varphi = \partial_\nu \left(-\theta_{||}^{\mu\nu} + \delta_{\mu 0} \varphi j_{||}^\nu - \frac{1}{2} g^{\mu\nu} \rho \varphi \right) \qquad (46)$$

It is easy to see that the expression (46) is correct by direct verification taking into account equality (42) and the formulae

$$\vec{j}_{||} = -\frac{1}{4\pi} \partial_t \vec{E}_{||}, \qquad \vec{E}_{||} = -\vec{\nabla}\varphi$$

Using Eqs. (39) and (46), one can derive the differential law of conservation

$$\partial_\nu T^{\mu\nu} = 0, \qquad (47)$$

where

$$T^{\mu\nu} = t^{\mu\nu} + \beta \theta_{||}^{\mu\nu} - \delta_{\mu 0} \beta \varphi j_{||}^\nu + \frac{1}{2}\beta \rho \varphi g^{\mu\nu} \qquad (48)$$

is the energy-momentum tensor of the system of interacting fields in nonrelativistic approximation.

To calculate the quantity $t^{\mu\nu}$ by the formula (40) it is convenient to write the function $\mathscr{L}'$ in the form:

$$\mathscr{L}' = \frac{1}{2}\sum_{\alpha=1,2}\left(\tilde{\phi}_\alpha^* \, i\vec{\partial}_t \phi_\alpha + \phi_\alpha^* \, i\vec{\partial}_t \tilde{\phi}_\alpha \right) - \frac{1}{2M}\sum_{\alpha=1,2}\left[\left(\vec{\pi}\tilde{\phi}_\alpha\right)^* \left(\vec{\pi}\phi_\alpha\right) + \left(\vec{\pi}\phi_\alpha\right)^* \left(\vec{\pi}\tilde{\phi}_\alpha\right) \right] + \qquad (49)$$

$$+ \frac{e\beta}{4M}\sum_{\alpha\beta=1,2}\left\{ \tilde{\phi}_\alpha^* \vec{\sigma}_{\alpha\beta}\left[\vec{\nabla}\left(\vec{\mathscr{A}}_\perp + \tilde{\vec{\mathscr{A}}}_\perp \right) \right]\phi_\beta + \phi_\alpha^* \vec{\sigma}_{\alpha\beta}\left[\vec{\nabla}\left(\vec{\mathscr{A}}_\perp + \tilde{\vec{\mathscr{A}}}_\perp \right) \right]\tilde{\phi}_\beta \right\} + \frac{\beta}{8\pi}\left(\partial^\alpha \tilde{\vec{\mathscr{A}}}_\perp \right)\left(\partial_\alpha \vec{\mathscr{A}}_\perp \right)$$

Herefrom

$$\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi_\alpha)} = \frac{i}{2}\begin{cases} \tilde{\phi}^*_\alpha, & \nu = 0 \\ \frac{1}{M}\left(-i\partial^\nu - e\beta A^\nu\right)\tilde{\phi}^*_\alpha, & \nu = 1,2,3; \end{cases} \qquad (50)$$

$$\frac{\partial \mathscr{L}}{\partial\left(\partial_\nu \vec{\mathscr{A}}_\perp\right)}\partial^\mu \vec{\mathscr{A}}_\perp = \frac{\beta}{8\pi}\begin{cases} \left(\partial^\nu \vec{\tilde{\mathscr{A}}}_\perp\right)\left(\partial^\mu \vec{\mathscr{A}}_\perp\right), & \nu = 0 \\ \left(\partial^\nu \vec{\tilde{\mathscr{A}}}_\perp\right)\left(\partial^\mu \vec{\mathscr{A}}_\perp\right) - \frac{e2\pi}{M}\left(\tilde{\phi}^+\left[\vec{\sigma}\partial^\mu \vec{\mathscr{A}}_\perp\right]_\nu \phi + \phi^+\left[\vec{\sigma}\partial^\mu \vec{\mathscr{A}}_\perp\right]_\nu \tilde{\phi}\right), & \nu = 1,2,3 \end{cases} \qquad (51)$$

Taking into account Eqs. (50) and (51), one can rewrite the formula (40 0 as follows;

$$t^{\mu 0} = \frac{i}{2}\left(\tilde{\phi}^+\vec{\partial}^\mu \phi + \phi^+\vec{\partial}^\mu \tilde{\phi}\right) + \frac{\beta}{8\pi}\left[\left(\partial_t \vec{\tilde{\mathscr{A}}}_\perp\right)\left(\partial^\mu \vec{\mathscr{A}}_\perp\right) + \left(\partial_t \vec{\mathscr{A}}_\perp\right)\left(\partial^\mu \vec{\tilde{\mathscr{A}}}_\perp\right)\right] - g^{\mu 0}\mathscr{L}; \qquad (52)$$

$$\begin{aligned} t^{\mu\nu} = {} & \frac{1}{2M}\left[\left(\partial^\nu \tilde{\phi}^+\right)\left(\partial^\mu \phi\right) + \left(\partial^\nu \phi^+\right)\left(\partial^\mu \tilde{\phi}\right) + \left(\partial^\mu \phi^+\right)\left(\partial^\nu \tilde{\phi}\right) + \left(\partial^\mu \tilde{\phi}^+\right)\left(\partial^\nu \phi\right) - \right. \\ & \left. - e\beta A^\nu\left(\tilde{\phi}^+ i\vec{\partial}^\mu \phi + \phi^+ i\vec{\partial}^\mu \tilde{\phi}\right)\right] + \frac{\beta}{8\pi}\left[\left(\partial^\nu \vec{\tilde{\mathscr{A}}}_\perp\right)\left(\partial^\mu \vec{\mathscr{A}}_\perp\right) + \left(\partial^\nu \vec{\mathscr{A}}_\perp\right)\left(\partial^\mu \vec{\tilde{\mathscr{A}}}_\perp\right)\right] - \\ & - \frac{e\beta}{4M}\left\{\tilde{\phi}^+\left[\vec{\sigma}\partial^\mu\left(\vec{\mathscr{A}}_\perp + \vec{\tilde{\mathscr{A}}}_\perp\right)\right]^\nu \phi + \phi^+\left[\vec{\sigma}\partial^\mu\left(\vec{\mathscr{A}}_\perp + \vec{\tilde{\mathscr{A}}}_\perp\right)\right]^\nu \tilde{\phi}\right\} - g^{\mu\nu}\mathscr{L}; \quad \nu = 1,2,3 \end{aligned}$$

Formula (47) leads to the integral law of conservation:

$$\int d\vec{r}\, T^{\mu 0} = \left(P^0, \vec{P}\right) = const \qquad (53)$$

Using the relations (42),(48) and (52), one comes to the formula for the energy density:

$$\begin{aligned} T^{00} = {} & \frac{\beta}{8\pi}\sum_\alpha \left(\partial^\alpha \vec{\tilde{\mathscr{A}}}_\perp\right)\left(\partial^\alpha \vec{\mathscr{A}}_\perp\right) + \frac{1}{2M}\left[\left(\vec{\pi}\tilde{\phi}\right)^+\left(\vec{\pi}\phi\right) + c.c.\right] - \\ & - \frac{e\beta}{2M}\left(\tilde{\phi}^+\,\vec{\sigma}\vec{H}\,\phi + \phi^+\,\vec{\sigma}\vec{H}\,\tilde{\phi}\right) + \frac{\beta}{8\pi}\vec{E}_{||}^{\,2} \end{aligned}$$

Taking into account that

$$\frac{\beta}{8\pi}\int d\vec{r}\,\vec{E}_{||}^{\,2} = H_{\text{int},1} \equiv W, \qquad (54)$$

one obtains the law of energy conservation (see the relations (25), (27) and (33)):

$$P^0 = \int d\vec{r}\, T^{00} = H = const$$

In the absence of the vortex electromagnetic field the energy of the electron field is

$$P^0 = \frac{1}{2M}\int d\vec{r}\left[\tilde{\phi}^+\left(-\vec{\nabla}^2\right)\phi + \phi^+\left(-\vec{\nabla}^2\right)\phi\right] + \frac{\beta}{8\pi}\int d\vec{r}\,\vec{E}_{||}^{\,2} \qquad (55)$$

Let us apply this formula to a stationary state of electron

$$\phi(\vec{r},t) = e^{-iEt}\psi(\vec{r}); \qquad \tilde{\phi}(\vec{r},t) = e^{-iEt}\tilde{\psi}(\vec{r}) \qquad (56)$$

where the components of the wave function ψ and $\tilde{\psi}$ obey the stationary Schrödinger equation

$$\begin{aligned} & \left(-\frac{\vec{\nabla}^2}{2M} + U(\vec{r})\right)\begin{pmatrix}\psi \\ \tilde{\psi}\end{pmatrix} = E\begin{pmatrix}\psi \\ \tilde{\psi}\end{pmatrix}, \qquad (57) \\ & U(\vec{r}) = e^2\beta\int d\vec{r}\,'\left|\vec{r} - \vec{r}\,'\right|^{-1}\left(\tilde{\psi}^+(\vec{r}\,')\psi(\vec{r}\,') + \psi^+(\vec{r}\,')\tilde{\psi}(\vec{r}\,')\right) \end{aligned}$$

Substituting (56) into (55) and using the equalities (54) and (57), one can obtain

$$P^0 = N\,E - W \qquad (58)$$

where

$$N = \int d\vec{r}\left(\tilde{\psi}^{+}\psi + c.c.\right) = -1$$

The momentum density is expressed by the formula

$$T^{\mu 0} = \frac{i}{2}\left(\tilde{\phi}^{+}\,\vec{\partial}^{\mu}\phi + \phi^{+}\,\vec{\partial}^{\mu}\tilde{\phi}\right) +$$

$$+ \frac{\beta}{8\pi}\left[\left(\partial_t \tilde{\vec{\mathcal{A}}}_{\perp}\right)\left(\partial^{\mu}\vec{\mathcal{A}}_{\perp}\right) + \left(\partial_t \vec{\mathcal{A}}_{\perp}\right)\left(\partial^{\mu}\tilde{\vec{\mathcal{A}}}_{\perp}\right)\right], \quad \mu = 1,2,3$$

Herefrom

$$\vec{P} = \vec{P}_{||} + \vec{P}_{\perp}, \qquad (59)$$

$$P_{||k} = \frac{1}{2}\int d\vec{r}\left[\tilde{\phi}^{+}\left(-i\vec{\nabla}_k\right)\phi + \phi^{+}\left(-i\vec{\nabla}_k\right)\tilde{\phi}\right]; \quad k = 1,2,3$$

$$P_{\perp k} = \frac{\beta}{8\pi}\int d\vec{r}\left[\left(\partial_t \tilde{\vec{\mathcal{A}}}_{\perp}\right)\left(-\nabla_k \vec{\mathcal{A}}_{\perp}\right) + \left(\partial_t \vec{\mathcal{A}}_{\perp}\right)\left(-\nabla_k \tilde{\vec{\mathcal{A}}}_{\perp}\right)\right]$$

where $\vec{P}_{||}$ and $\vec{P}_{\perp}$ are the momenta of the electron and the vortex electromagnetic field, respectively. Taking into account the Fourier expansion

$$\left\{\vec{\mathcal{A}}_{\perp}(x); \tilde{\vec{\mathcal{A}}}_{\perp}(x)\right\} = V^{-1/2}\sum_{\vec{k}}\sum_{\lambda=1,2}\left\{\mathcal{A}_{\vec{k}\lambda}; \tilde{\mathcal{A}}_{\vec{k}\lambda}\right\}\vec{e}_{\vec{k}\lambda}\, e^{i\vec{k}\vec{r}}, \qquad (60)$$

where V is the normalization volume; $\vec{e}_{\vec{k}1}$, $\vec{e}_{\vec{k}2}$, $\vec{e}_{\vec{k}3} = \dfrac{\vec{k}}{k}$ are the mutually orthogonal vectors satisfying the relations

$$\vec{e}^{\,*}_{\vec{k}\lambda}\vec{e}_{\vec{k}\lambda'} = \delta_{\lambda\lambda'}; \qquad \vec{e}^{\,*}_{-\vec{k}\lambda} = (-1)^{\lambda}\vec{e}_{\vec{k}\lambda},$$

one finds

$$\vec{P}_{\perp} = \frac{i\beta}{8\pi}\sum_{\vec{k}}\sum_{\lambda=1,2}(-1)^{\lambda}\vec{k}\left(\tilde{\mathcal{A}}_{\vec{k}\lambda}\mathcal{A}_{-\vec{k}\lambda} + \dot{\mathcal{A}}_{\vec{k}\lambda}\tilde{\mathcal{A}}_{-\vec{k}\lambda}\right) \qquad (61)$$

Note the following representation

$$\vec{P}_{\perp} = \frac{\beta}{8\pi}\int d\vec{r}\left(\tilde{\vec{E}}_{\perp}\times\vec{B} + \vec{E}_{\perp}\times\tilde{\vec{B}}\right) \qquad (62)$$

where

$$\vec{B} = \left[\vec{\nabla}\times\vec{\mathcal{A}}_{\perp}\right], \qquad \tilde{\vec{B}} = \left[\vec{\nabla}\times\tilde{\vec{\mathcal{A}}}_{\perp}\right]$$

For a nonstationary state (A. 11) the quantities P^0 (55) and $\vec{P}_{||}$ (59) are

$$P^0 = N\left(E + \frac{1}{2}M\vec{v}_0^{\,2}\right) - W, \qquad \vec{P}_{||} = NM\vec{v}_0$$

Since the wave function (A. 11) describes the electron moving in the space as a whole with the velocity $\vec{v}_0$, the formulae obtained above can be easily interpreted in the following way. By virtue of the fact that the normalization integral for the self-acting electron is negative $(N = -1)$, the part of physical energy and momentum is played by the quantities

$$-P^0 = \frac{1}{2}M\vec{v}_0^{\,2} + W + E \qquad \text{and} \qquad -\vec{P}_{||} = M\vec{v}_0 \qquad (63)$$

The quantities $\dfrac{1}{2}M\vec{v}_0^{\,2}$, W and E have the physical meaning, respectively, of kinetic energy of motion of the electron as a whole, potential energy of self-action and energy of the electron in the field of potential well, appearing as a result of Coulomb interaction.

4. The Ehrenfest theorem. Force of the self-action

Let us consider the generalization of the Ehrenfest theorem for the self-acting electron [9] to the case when the particle spin is taken into account. The mean values of position vector and momentum vector of the electron in an external field $(j, \vec{A})$ in the state f are defined by the formulae

$$\overline{\vec{r}} = \int d\vec{r}\, \phi^+ \vec{r}\, \phi\,; \qquad \overline{\vec{p}} = \int d\vec{r}\, \phi^+ \left(-i\vec{\nabla} - e\vec{A}\right)\phi \qquad (64)$$

where the wave function $\phi = \phi(\vec{r}, t)$ obeys the equation

$$i\frac{\partial\phi}{\partial t} = \left\{ \frac{1}{2M}\left(-i\vec{\nabla} - eA\right)^2 - \frac{e}{2M}\vec{\sigma}\vec{H} + U + e\varphi \right\}\phi, \qquad (65)$$

$\vec{H} = rot\vec{A}$, U is the potential energy of the self-action, defined by the formula (11). Using the equation (65), one can get

$$\frac{d\overline{\vec{r}}}{dt} = \frac{\overline{\vec{p}}}{M}; \qquad (66)$$

$$\frac{d\overline{\vec{p}}}{dt} = \int d\vec{r}\, \phi^+\phi\left(e\vec{E} - \frac{ie}{2M}\left[\vec{\nabla}\times\vec{H}\right]\right) - e\int d\vec{r}\, \phi^+ \left\{\left[\vec{H}\times\hat{\vec{v}}\right] - \frac{1}{2M}\overset{\mapsto}{\vec{\nabla}}(\vec{\sigma}\vec{H})\right\}\phi$$

where $\vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial\vec{A}}{\partial t}$, $\qquad \hat{\vec{v}} = \dfrac{\vec{p}}{M} = \left(-i\vec{\nabla} - e\vec{A}\right)/M\,; \qquad \vec{\nabla}\vec{A} = 0.$

Combining the relations (66), one can obtain the quantum Newton equations:

$$M\frac{d^2\overline{\vec{r}}}{dt^2} = \int d\vec{r}\, \phi^+ \left\{ e\vec{E} - e\left[\vec{H}\times\hat{\vec{v}}\right] - \frac{ie}{2M}\left[\overset{\mapsto}{\vec{\nabla}}\times\vec{H}\right] + \frac{e}{2M}\overset{\mapsto}{\vec{\nabla}}(\vec{\sigma}\vec{H}) \right\}\phi \qquad (67)$$

As the equation (67) illustrates, the quantum equation of motion depends on the electron spin only in the case of inhomogeneous magnetic field. If the external field is homogeneous, the equation (67) reduces to the classical equation of motion for the center of mass of the electron

$$M\frac{d^2\overline{\vec{r}}}{dt^2} = e\vec{E} + e\left[\overline{\vec{v}}\times\vec{H}\right] \qquad (68)$$

According to (67), as for the spinless particle, the center of mass of the nonrelativistic electron moves in such a way as if the force of the Coulomb self-action were absent.

Let us turn our attention now to the relativistic self-acting electron. Eliminating in the equation (1) the components of the vortex electromagnetic field with the aid of Maxwell's equations and taking into account the external field $A_{ext} = \left(A^0_{ext}, \vec{A}_{ext}\right)$, we arrive at the following equation of motion:

$$\left(i\hat{\partial} - \hat{\mathscr{A}}(x) - M\right)\begin{pmatrix}\psi(x)\\ \tilde{\psi}(x)\end{pmatrix} = 0 \qquad (69)$$

where

$$\mathscr{A}^\mu(x) = e\beta A^\mu(x) + e\,A^\mu_{ext}(x),$$
$$A^\mu(x) = \int d^4x'\,\delta\left((x-x')^2\right) j^\mu(x')$$

Denoting the mean value of the operator $\hat{F}$ in the state ψ by $\overline{F} = \int d\vec{r}\,\psi^+\hat{F}\psi$ and using the equation (69), it is easy to derive the relation

$$\frac{d\overline{F}}{dt} = \int d\vec{r}\,\psi^+\left\{ i\left[\vec{\alpha}\left(-i\frac{\partial}{\partial\vec{r}} - \vec{\mathscr{A}}\right) + \mathscr{A}^0 + \gamma^0 M\right]\hat{F} - \right.$$
$$\left. - i\hat{F}\left[\vec{\alpha}\left(-i\frac{\partial}{\partial\vec{r}} - \vec{\mathscr{A}}\right) + \mathscr{A}^0 + \gamma^0 M\right] + \frac{\partial\hat{F}}{\partial t}\right\}\psi$$

Putting in the last equality $\hat{F} = \vec{r}$ and $\hat{F} = \vec{p} - \vec{\mathscr{A}}$, $\vec{p} = -i\dfrac{\partial}{\partial \vec{r}}$, one obtains the generalization of the Ehrenfest theorem in the relativistic case [22] (cf. (66)).

$$\frac{d\,\overline{\vec{r}}}{d\,t} = \int d\vec{r}\,\psi^{+}\vec{\alpha}\psi \equiv \overline{\vec{\alpha}}; \qquad (70)$$

$$\frac{d}{d\,t}\overline{\left(\vec{p}-\vec{\mathscr{A}}\right)} = \int d\vec{r}\,\psi^{+}\left(\vec{F}+\vec{F}_{ext}\right)\psi \equiv \overline{\vec{F}} + \overline{\vec{F}}_{ext}$$

where

$$\vec{F} = e\beta\left(\vec{E} + \vec{\alpha}\times\vec{B}\right),$$

$$\vec{F}_{ext} = e\left(\vec{E}_{ext} + \vec{\alpha}\times\vec{B}_{ext}\right),$$

$$\vec{E} = -\frac{\partial\,\vec{A}}{\partial\,t} - \vec{\nabla}A^{0}, \qquad \vec{B} = \vec{\nabla}\times\vec{A},$$

$$\vec{E}_{ext} = -\frac{\partial\,\vec{A}_{ext}}{\partial\,t} - \vec{\nabla}A^{0}_{ext}, \quad \vec{B}_{ext} = \vec{\nabla}\times\vec{A}_{ext}$$

$\vec{F}$ is the self-acting force, $\vec{F}_{ext}$ is the force experienced by the particle due to the external field. Since the value $\overline{\vec{\alpha}}$ differs from $\overline{\left(\vec{p}-\vec{\mathscr{A}}\right)}\big/M$, it is impossible to obtain from Eqs. (70) the relativistic equation of motion for the center of mass of the particle.

To estimate the value of the mean self-action force $\vec{F}$ it is convenient to use as the units of length, energy and time the values a_0, I_0 and I_0^{-1}, respectively, where $a_0 = 1/Me^2$ is the Bohr radius, $I_0 = Me^4/2$ is the ionization energy of hydrogen atom. Let us introduce the notation:

$$t = t'I_0^{-1}, \qquad \vec{r} = \vec{r}'a_0, \qquad \psi(x)a_0^{3/2} = \psi'(x')$$

$$I_0^{-1}eA^{\mu}(x) = A'^{\mu}(x') \qquad (71)$$

where $x' = (t', \vec{r}')$; t' and $\vec{r}'$ are the dimensionless time and position vector. In dimensionless notation the quantity $\overline{\vec{F}}$ can be written as follows ($I_0 a_0^{-1}$ is the unit of force):

$$\overline{\vec{F}} = I_0\,a_0^{-1}\beta\int d\vec{r}'\psi'^{+}(x')\left\{-\frac{\alpha}{2}\frac{\partial}{\partial t'}\vec{A}'(x') - \right.$$

$$\left. -\frac{\partial}{\partial \vec{r}'}A'^{0}(x') + \vec{\alpha}\times\frac{\partial}{\partial \vec{r}'}\times\vec{A}'(x')\right\}\psi'(x') \qquad (72)$$

Let us expand the function $A'^{\mu}(x')$ in a power series in the fine structure constant α:

$$A'^{\mu}(x') = 2N\int d\vec{r}_1'\frac{j'^{\mu}(t', \vec{r}_1')}{\left|\vec{r}' - \vec{r}_1'\right|} + \frac{\alpha^2 N}{4}\int d\vec{r}_1'\left|\vec{r}' - \vec{r}_1'\right|\frac{\partial^2}{\partial t'^2}j'^{\mu}(t', \vec{r}_1') + \ldots \qquad (73)$$

Here

$$j'^{\mu}(t, \vec{r}) = \overline{\psi}'(x)\gamma^{\mu}\psi'(x)$$

Further, substituting (73) into (72) and keeping only the largest in magnitude terms, one can obtain

$$\overline{\vec{F}} = -I_0\,a_0^{-1}\beta\,N\alpha\int d\vec{r}'\int d\vec{r}_1'\left|\vec{r}' - \vec{r}_1'\right|^{-1}\left\{\frac{\partial}{\partial t'}\left[j'^{0}(x')\,\vec{j}'(t', \vec{r}_1')\right] + \right.$$

$$\left. + \frac{\alpha}{4}\left(\vec{r}' - \vec{r}_1'\right)j'^{0}(x')\frac{\partial^2}{\partial t'^2}j'^{0}(t', \vec{r}_1')\right\} \qquad (74)$$

While deriving this formula the continuity equation $\vec{\nabla}'\vec{j}'+\frac{\alpha}{2}\frac{\partial}{\partial t'}j'^{0}=0$ has been used. As is seen from Eq. (74) $\bar{\bar{F}}\approx\alpha^2$ (as $\vec{j}'(x')\approx\alpha$), with $\bar{\bar{F}}=0$ for the stationary electron state (as in the stationary state $\frac{\partial}{\partial t'}j'^{\mu}(x')=0$). Thus, the formula (74) confirms the conclusion made in [9] that in the nonrelativistic approximation, taking into account only the α-order corrections, the mean force of the self-action is absent.

5. Non-stationary electron states in the homogenous external field

Let us consider the self-acting electron in the external field described by the potentials

$$\varphi=-\vec{E}\vec{r}\,,\ \vec{A}=\frac{1}{2}\left[\vec{H}\times\vec{r}\right] \qquad (75)$$

where $\vec{E}=(0,E_y,E_z)$, $\vec{H}=(0,0,H)=const$, $E_y=E_y(t)$ and $E_z=E_z(t)$ are the arbitrary functions of time. Taking into account the results of the previous section, it is natural to look for the solution of the equation (65) in the form

$$\phi=u_\sigma\exp(-i\chi_\sigma(t,\vec{r}))\Psi\left(\vec{r}-\vec{R}_0(t)\right)\equiv\phi_\sigma(\vec{r},t),\ \sigma=\pm1 \qquad (76)$$

where u_s is the spinor: $u_1\begin{pmatrix}1\\0\end{pmatrix}$, $u_{-1}\begin{pmatrix}0\\1\end{pmatrix}$; $\vec{R}_0=\vec{R}_0(t)$ is the solution of the classical equation of motion

$$M\ddot{\vec{R}}_0=e\vec{E}+e\left[\dot{\vec{R}}_0\times\vec{H}\right], \qquad (77)$$

$\chi_\sigma(t,\vec{r})$ and $\Psi(\vec{r})$ are the functions to be determined. Substituting (76) into (65) yields

$$\left\{\frac{\partial\chi_\sigma}{\partial t}-\left[\dot{\vec{R}}_0+\frac{1}{M}\left(\vec{\nabla}\chi_\sigma+\frac{e}{2}\left[\vec{H}\times\vec{R}_0\right]\right)\right]i\vec{\nabla}_{\vec{R}}-\right. \qquad (78)$$

$$-\frac{1}{2M}\left[\left(-i\vec{\nabla}_{\vec{R}}-e\vec{A}'\right)^2+2e\vec{A}'\left(\vec{\nabla}\chi_\sigma+\frac{e}{2}\left[\vec{H}\times\vec{R}_0\right]\right)+\left(\vec{\nabla}\chi_\sigma+\right.\right.$$

$$\left.\left.+\frac{e}{2}\left[\vec{H}\times\vec{R}_0\right]\right)^2+i\vec{\nabla}^2\chi_\sigma\right]+e\vec{E}\left(\vec{R}+\vec{R}_0\right)-U\left(\vec{R}\right)+\frac{e\sigma H}{2M}\Bigg\}\Psi\left(\vec{R}\right)=0$$

The notation

$$\vec{r}-\vec{R}_0=\vec{R},\qquad \vec{A}'=\frac{1}{2}\left[\vec{H}\times\vec{R}\right], \qquad (79)$$

$$U\left(\vec{R}\right)=\beta e^2N\int d\vec{R}'\left|\vec{R}-\vec{R}'\right|^{-1}\left|\Psi\left(\vec{R}'\right)\right|^2$$

was used above.

One can determine the function χ_σ from the equation

$$\vec{\nabla}\chi_\sigma+\frac{e}{2}\left[\vec{H}\times\vec{R}_0\right]=-M\dot{\vec{R}}_0\equiv-\vec{P}_0 \qquad (80)$$

The solution of this equation can be written in the form

$$\chi_\sigma(t,\vec{r})=-\frac{e}{2}\left(\vec{r}\left[\vec{H}\times\vec{R}_0\right]\right)-\vec{r}\vec{P}_0+f_\sigma(t) \qquad (81)$$

where the function $f_\sigma(t)$ will be defined later. Taking into account (81) and (77), one can derive

$$\vec{\nabla}^2\chi_\sigma = 0$$
$$\frac{\partial\chi_\sigma}{\partial t} = -\frac{e}{2}\left(\vec{r}\left[\vec{H}\times\vec{V}_0\right]\right) - \left(e\vec{E} + e\left[\vec{V}_0\times\vec{H}\right]\right)\vec{r} + \frac{\partial f_\sigma}{\partial t} \qquad (82)$$

where $\vec{V}_0 = \dot{\vec{R}}_0$. By virtue of Eqs. (80)–(82) the equation can be transformed to the form:

$$\left\{\frac{\partial f_\sigma}{\partial t} - \frac{e}{2}\left(\vec{R}_0\left[\vec{V}_0\times\vec{H}\right]\right) - \frac{\vec{P}_0^2}{2M} + \frac{e\sigma H}{2M} - \frac{1}{2M}\left(-i\vec{\nabla}_{\vec{R}} - e\vec{A}'\right)^2 - U\left(\vec{R}\right)\right\}\Psi\left(\vec{R}\right) = 0$$

Defining the function f_σ by the equality

$$\frac{\partial f_\sigma}{\partial t} - \frac{e}{2}\left(\vec{R}_0\left[\vec{V}_0\times\vec{H}\right]\right) - \frac{\vec{P}_0^{\,2}}{2M} + \frac{e\sigma H}{2M} = E = const\,, \qquad (83)$$

one can obtain the following equation for the function $\Psi\left(\vec{R}\right)$:

$$\left[\frac{1}{2M}\left(-i\vec{\nabla}_{\vec{R}} - e\vec{A}'\right)^2 + U\left(\vec{R}\right) - E\right]\Psi\left(\vec{R}\right) = 0 \qquad (84)$$

From (83) we obtain the following expression for the function $f_\sigma(t) = f_\sigma(t)$:

$$f_\sigma(t) = \frac{1}{2M}\int_0^t dt'\left[\vec{P}_0^{\,2}(t') + e\left(\vec{R}_0(t')\left[\vec{P}_0(t')\times\vec{H}\right]\right)\right] + \left(E - \frac{e\sigma H}{2M}\right)t \qquad (85)$$

The general solution of Eq. (77) can be represented in the form (see [23]):

$$\vec{R}_0(t) = \left(R_{0x}, R_{0y}, R_{0z}\right), \qquad (86)$$
$$R_{0x} = \frac{v_{0x}}{\omega_{0x}}\sin\omega_0 t + \frac{v_{0y}}{\omega_0}(1-\cos\omega_0 t) + \frac{e}{M}\int_0^t dt'\int_0^{t'} dt''\sin\left[\omega_0(t'-t'')\right]E_y(t'') + x_0\,;$$
$$R_{0y} = -\frac{v_{0x}}{\omega_0}(1-\cos\omega_0 t) + \frac{v_{0y}}{\omega_0}\sin\omega_0 t + \frac{e}{M}\int_0^t dt'\int_0^{t'} dt''\cos\left[\omega_0(t'-t'')\right]E_y(t'') + y_0\,;$$
$$R_{0z} = \frac{e}{M}\int_0^t dt'\int_0^{t'} dt''E_z(t'') + v_{0z}t + z_0\,, \qquad \omega_0 = \frac{eH}{M}.$$

The vectors $\vec{R}_0(t)$ and $\vec{V}_0(t) = \dot{\vec{R}}_0(t)$ obey the initial conditions:

$$\vec{R}_0(0) = (x_0, y_0, z_0) \equiv \vec{r}_0 = const$$
$$\vec{V}_0(0) = \left(v_{0x}, v_{0y}, v_{0z}\right) \equiv \vec{v}_0 = const$$

Equalities (76), (81), (85) and (86) and equation (84) completely determine the electron wave function representing the non-stationary state in the homogeneous external field.

Note that the linear combinations of the functions ϕ_σ (76) of the form $\sum_{\sigma=\pm1} C_\sigma\phi_\sigma(\vec{r},t)$ (C_σ are constants satisfying the condition $|C_1|^2 + |C_{-1}|^2 = 1$) are also the solutions of Eq. (65).

At $\vec{H} = 0,\ \vec{E} = const$ and at $\vec{H} \neq 0,\ \vec{E} = 0$ the results of the paper [9] may be obtained. In particular, at $\vec{H} = 0,\ \vec{E} = const$ the formulae (81), (85) and (86) yields:

$$\vec{R}_0(t) = \frac{e\vec{E}}{2M}t^2 + \vec{v}_0 t + \vec{r}_0\,,$$
$$f_\sigma(t) = \frac{e^2\vec{E}^{\,2}}{6M}t^3 + \frac{e\vec{E}\,\vec{v}_0}{2}t^2 + \left(E + \frac{M\,v_0^2}{2}\right)t\,,$$
$$\chi_\sigma(t,\vec{r}) = -e\vec{E}\,\vec{r}\,t - M\,\vec{v}_0\vec{r} + f_\sigma(t),$$

which agrees with the results of the paper [9].

The charge and current densities of the electron in the state $\phi_\sigma(\vec{r},t)$ (76) are given by (we use the relationships (10) and (16) and assume the equalities $\tilde{\phi}_\sigma = a\phi$, $a = const$, $a + a^* = N$ to be fulfilled):

$$\rho_\sigma = eN|\Psi|^2, \qquad (87)$$

$$\vec{j}_\sigma = eN\left\{\vec{V}_0 - \frac{e}{2M}[\vec{H}\times\vec{R}] + \frac{\sigma}{2M}\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}, 0\right)\right\}|\Psi|^2 - \frac{ieN}{2M}\Psi^*\vec{\vec{\nabla}}\Psi, \quad \sigma = \pm 1$$

where $\Psi = \Psi(\vec{R})$, $\vec{R} = \vec{r} - \vec{R}_0$. It is seen from Eq. (87) that for $\vec{E} = \vec{v}_0 = 0$ the wave function (76) describes a stationary state in which the current becomes purely vortex ($\vec{j}_\sigma = \vec{j}_{\sigma\perp}$).

6. The free electron

Let us consider the solutions of Eqs. (12)describing the stationary states (56) of the free electron. Restricting ourselves to the spherically symmetric solutions and using the constraint (3) and the notation

$$\Psi(r) = \frac{1}{r}X(r), \qquad U(r) = U(0) + \frac{1}{r}Z(r) \qquad (88)$$

we can obtain the following set of differential equations of the second order which is equivalent to the equations (57) (in dimensional form, see [6]):

$$\left(\frac{d^2}{dr^2} - \frac{Z}{r} + C\right)X = 0, \qquad (89)$$

$$\frac{d^2}{dr^2}Z = -8\pi\beta N\frac{1}{r}X^2, \qquad C = E - U(0)$$

Here

$$Z(r) = 8\pi\beta N\int_0^r dr'\left(1 - \frac{r}{r'}\right)X^2(r'),$$

$$U(0) = 8\pi\beta N\int_0^\infty dr\, r^{-1}X^2(r) \qquad (90)$$

The sought-for solution of the set of Eqs. (89) is subject to the normalisation condition

$$4\pi\int_0^\infty drX^2(r) = 1 \qquad (91)$$

and may be expanded in a power series in the vicinity of $r = 0$ $(a = const)$

$$X = ar(1 + a_1r^2 + \ldots),$$

$$Z = -\frac{4\pi}{3}\beta Na^2r^3\left(1 + \frac{3}{5}a_1r^2 + \ldots\right), \quad a_1 = -\frac{C}{6}$$

The potential particle energy W (see (21)) and its total energy $\mathscr{E}$ (see (63)) are expressed by the equalities

$$W = \frac{N}{2}\int d\vec{r}\,U(r)|\Psi(\vec{r})|^2 = N\left(\frac{1}{2}U(0) + 2\pi\int_0^\infty\frac{dr}{r}Z(r)X^2(r)\right) \qquad (92)$$

$$\mathscr{E} = E + W \qquad (93)$$

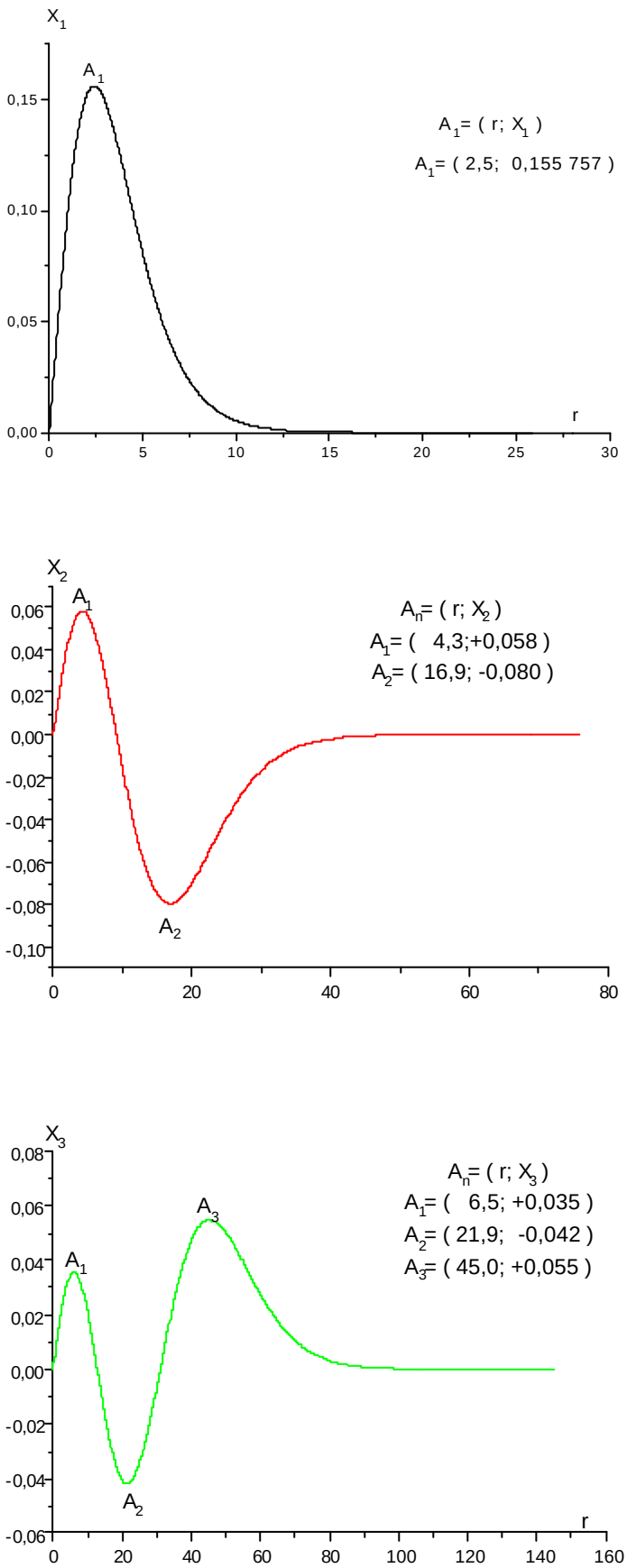

Fig.1 Wave functions of free electron $X_n = X_n (r)$ (n=1,2,3)
(the points $A_n = A_n (r; X)$ determine the position of extrema)

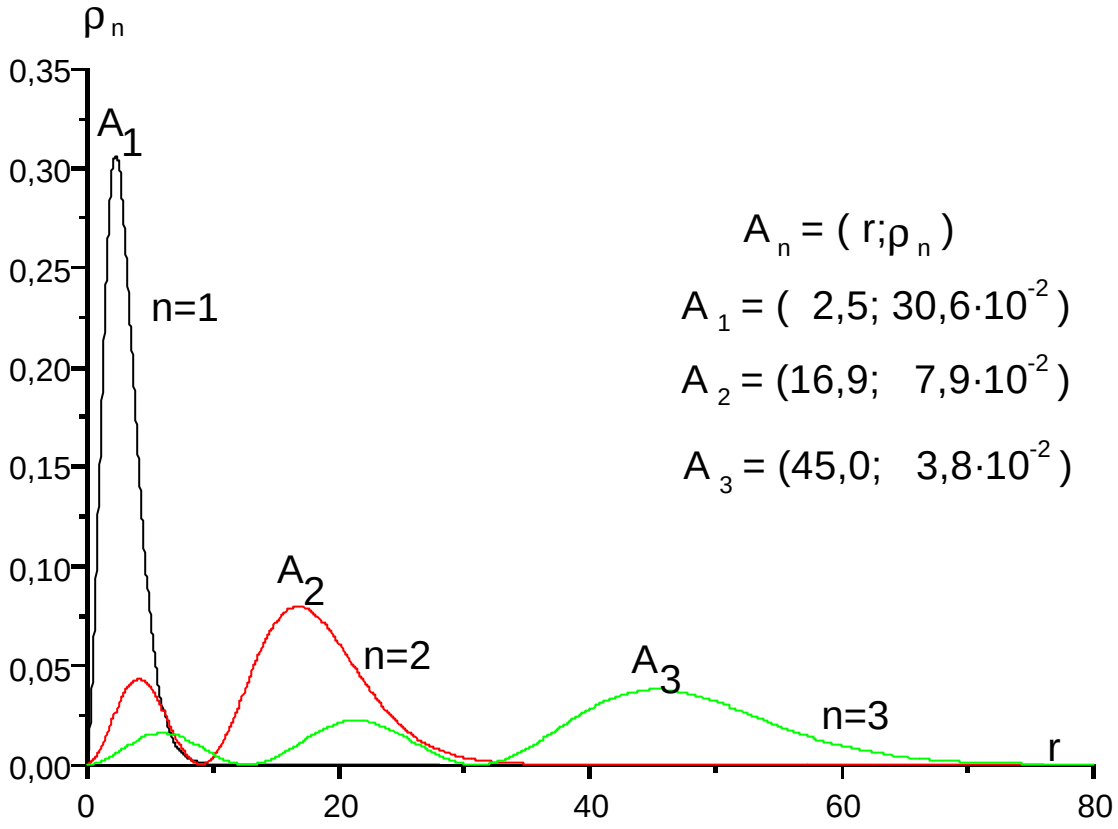

Fig.2 Density of particle number for a free electron (the points $A_n = (r; \rho_n)$ determine the position of principal maxima).

Table of the quantum state parameters of free electron

n	1	2	3
a_n	0 , 111 555 302	0 , 022 550 636	0 , 009 419 618
C_n	0 , 610 920 251	0 , 162 669 776	0 , 075 459 384
E_n	- 0 , 651 077	- 0 , 123 186	- 0 , 050 105
$\mathscr{E}_n$	- 0, 217 025	- 0 , 041 062	- 0 , 016 702

Figures 1 and 2 show the plots of the wave functions $X_n = X_n(r)$ and particle number densities $\rho_n(r) = 4\pi X_n^2(r)$ for the ground $(n = 1)$ and the first two excited $(n = 2,3)$ electron states. In the Table the magnitudes of the parameters a_n, C_n, E_n, and $\mathscr{E}_n$, corresponding to the particle states indicated above, are given (we put here $\beta = \sqrt{2}$).

7. The electron in constant electric field

Let us consider a stationary state of self-acting electron in electric field with the intensity $\vec{\mathscr{E}} = (0,0,\mathscr{E}')$, $\mathscr{E}' = const$. The perturbation theory for the Schrödinger equation describing the self-acting electron behaviour in external field is constructed in [9]. Here we are going to use the results of this paper, carrying out the calculations only in the first order of perturbation theory with respect to electric field. In this approximation only the first two spherical harmonics can be retained in the expansions of the functions $X(\vec{r})$ and $Z(\vec{r})$ (see formulae (52) [9]):

$$X(\vec{r}) = (4\pi)^{1/2} \sum_{l=0,1} X_l(r) Y_{l0}(\theta,\varphi), \qquad (94)$$

$$Z(\vec{r}) = (4\pi)^{1/2} \sum_{l=0,1} Z_l(r) Y_{l0}(\theta,\varphi)$$

The functions X_l and Z_l obey the set of equations (see Eqs. (55) [9])

$$\left(\frac{d^2}{dr^2} - \frac{Z_0}{r} + C\right) X_0 = \left[(C_1 + \varepsilon) r + \frac{Z_1}{r}\right] X_1;$$

$$\frac{d^2}{dr^2} Z_0 = -8\pi\beta N r^{-1}\left(X_0^2 + X_1^2\right); \qquad (95)$$

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2} - \frac{Z_0}{r} + C\right) X_1 = \left[(C_1 + \varepsilon) r + \frac{Z_1}{r}\right] X_0;$$

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) Z_1 = -16\pi\beta N r^{-1} X_0 X_1$$

Here

$$\varepsilon = \frac{e_0 \mathscr{E}' a_0}{I_0 \sqrt{3}}, \qquad e_0 = |e|, \qquad C = E - U(0)$$

$$U(0) = 8\pi\beta N \int_0^\infty \frac{dr}{r}\left(X_0^2(r) + X_1^2(r)\right) = C_0,$$

$$C_1 = \frac{16}{3}\pi\beta N \int_0^\infty \frac{dr}{r^2} X_0(r) X_1(r),$$

E is the energy eigenvalue of electron in the electric field. The set of equations (95) is valid provided the condition

$$\varepsilon R << |E_0| \qquad (96)$$

is fulfilled. In (96) R is the width of the localization region of the wave function in zero approximation, E_0 is the energy eigenvalue of the particle in the absence of electric field. The first two equations of the set (95), if we put $X_1 = 0$ in them, coincide with Eqs. (89); thus, they correspond to the zeroth approximation.

The desired solution of the equations (95) has to obey the conditions

$$4\pi \int_0^\infty dr\left(X_0^2(r) + X_1^2(r)\right) = 1, \qquad (97)$$

$$C_1 = -\frac{1}{3r^2}\left(Z_1(r) + r\frac{dZ_1(r)}{dr}\right)\Bigg|_{r\to\infty}$$

and behave, when $r \to 0$, in such a way (a, b are constants):

$$X_0 = ar\left(1 + a_1 r^2 + \ldots\right); \qquad X_1 = r^2\left(b + b_1 r^2 + \ldots\right)$$

$$Z_0 = -\frac{4\pi}{3}\beta N a^2 r^3 + ...; \qquad Z_1 = -\frac{8\pi}{5}\beta N a b r^4 + ...$$

where

$$a_1 = -\frac{C}{6}, \quad b_1 = \frac{1}{10}\left((C_1 + E)a - Cb\right)$$

One can see from Eqs. (94) and (97) in the approximation being considered the electron wave function in the electric field represents a superposition of the ground and the first harmonics, each of them being a solitons. If the notation

$$V_0 = \frac{Z_0}{r} + U(0), \qquad V_1 = \frac{2}{r^2} + V_0,$$
$$U_1 = \frac{Z_1}{r} + (C_1 + \varepsilon) r$$

is introduced, then the first and the third of the equations (95) can be written in the following simple form

$$\left(\frac{d^2}{dr^2} + E - V_0\right) X_0 = U_1 X_1, \qquad (99)$$
$$\left(\frac{d^2}{dr^2} + E - V_1\right) X_1 = U_1 X_0$$

Obviously, the quantities V_0 and V_1 play the part of the potential energy for solitons corresponding to the ground and the first harmonics, respectively.

The potential energy of self-action of electron W and the mean square of the orbital angular momentum L^2,

$$L^2 = \int d\vec{r}\, \Psi^*(\vec{r}) \hat{\vec{L}}^2 \Psi(\vec{r}),$$

can be transformed to the form

$$W = \frac{N}{2} \sum_{lm} \int_0^\infty dr \left(C_{lm} r^l + \frac{1}{r} Z_{lm}(r) \right) B^*_{lm}(r),$$
$$L^2 = \sum_{lm} l(l+1) \int_0^\infty dr \left| X_{lm}(r) \right|^2. \qquad (100)$$

Here we have assumed that $\Psi(\vec{r}) = X(\vec{r})/r$. The rest of notation is the same as in [9] (see formulae (29)-(32) [9]). In the approximation here considered as applied to electric field, in the right-hand sides of Eqs. (100) one has to retain only two addends: $l = m = 0$ and $l = 1$, $m = 0$ and take into account the equalities

$$B_{00} = (4\pi)^{1/2} \left(X_0^2 + X_1^2 \right); \qquad B_{10} = 4\pi^{1/2} X_0 X_1;$$
$$\left(X_{l0}; Z_{l0}; C_{l0} \right) = (4\pi)^{1/2} \left(X_l; Z_l; C_l \right), \qquad l = 0,1$$

As a result, one can obtain

$$W = W_0 + W_1;$$
$$W_0 = N \left[\frac{1}{2} U(0) + 2\pi \int_0^\infty dr \frac{1}{r} Z_0(r) \left(X_0^2(r) + X_1^2(r) \right) \right];$$

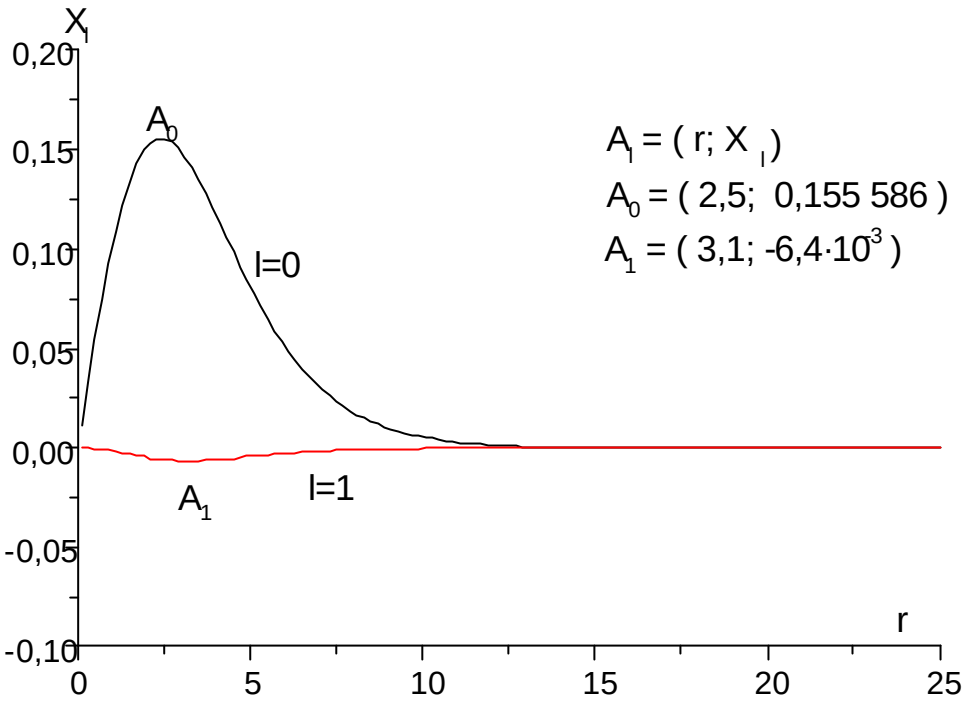

Fig. 3 Components of electron wave functions $X_l = X_l (r)$ (l=0, 1) in an electric field (the points $A_l = (r; X_l)$ determine the position of extrema)

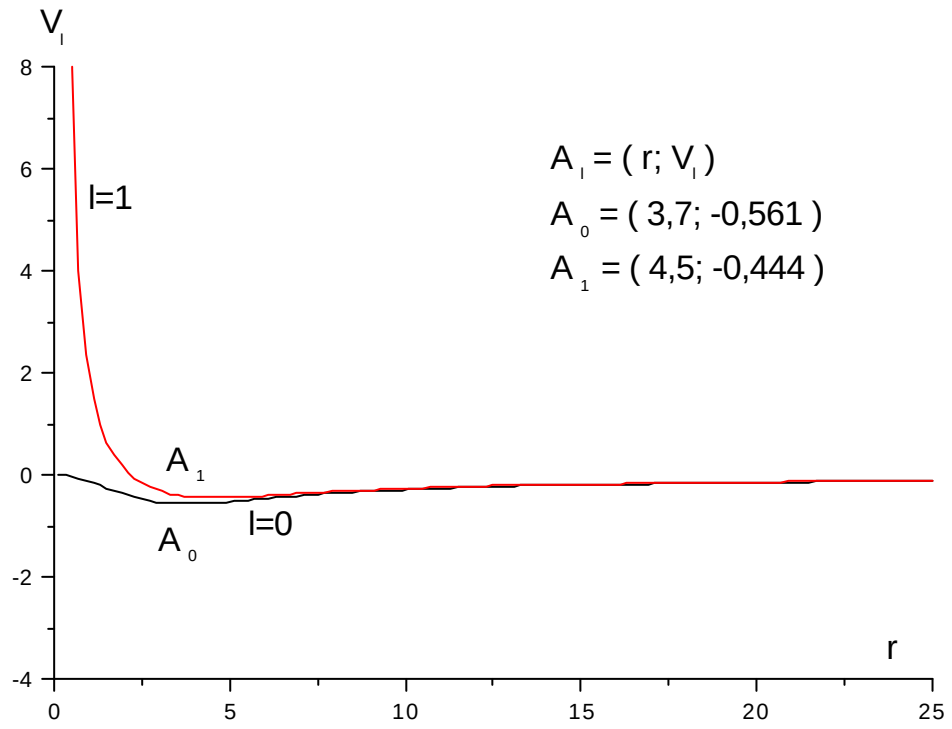

Fig. 4 Potential energy $V_l = V_l (r)$ (l=0,1) corresponding to the electron wave function harmonics in an electric field (the point $A_l = (r; V_l)$ determine the position of minima)

$$W_1 = N4\pi\int_0^\infty dr\left(C_1 r + \frac{Z_1(r)}{r}\right)X_0(r)X_1(r);$$

$$L^2 = 8\pi\int_0^\infty drX_1^2(r)$$

The total energy of the electron can be determined by formula (93).

The results of numerical calculation for the quantum state which in the absence of electric field goes over into the ground state of the free electron are shown in Fig. 3 and 4 (we assumed here that $\beta = \sqrt{2},\ \varepsilon = 10^{-5},\ r^* = 10^{-9},\ r^*$ is the initial magnitude of the variable r). Below are given the magnitudes of the parameters corresponding to the computer-calculated quantum state of the electron in electric field:

a = 0, 111 289 017	E = -0, 651 075
C = 0, 609 192 019	W_0 = 0, 433 518
b = -2, 768 575· 10^{-3}	W_1 = 5, 29· 10^{-4}
C_1 = 1, 297 792· 10^{-2}	$\mathscr{E}$ = -0, 217 028
	L = 5, 8· 10^{-2}

As evident from Fig. 1 and 3, the maximum height of the main harmonic of the wave function in the electric field is slightly smaller than of the wave function of the free electron.

8. The electron in constant magnetic field

The expansion of the electron wave function in magnetic field $\vec{\mathscr{H}} = (0,0,\mathscr{H}),\ \mathscr{H} = const$, in the series of perturbation theory with respect to magnetic field is given in [9]. Following are some refinements of the theory outlined in [9] and the results of the numerical calculations for the quantum state which in the absence of magnetic field goes over into the ground state of the free electron.

Let us write down the formulae for the Fourier coefficients $B_{\sigma l0}, A_{\sigma l0}, G_{\sigma l0}, F_{\sigma l0}, (l = 0,2)$ (see pages 22 and 23 [9]) in which the terms of higher order of smallness are taken into account:

$$\begin{aligned}
B_{\sigma 00} &= (4\pi)^{-1/2}\left(X_{\sigma 00}^2 + X_{\sigma 20}^2\right);\\
B_{\sigma 20} &= \pi^{-1/2}X_{\sigma 00}X_{\sigma 20} + \frac{1}{7}\sqrt{\frac{5}{\pi}}X_{\sigma 20}^2;\\
A_{\sigma 00} &= (4\pi)^{-1/2}\left(X_{\sigma 00}Z_{\sigma 00} + X_{\sigma 20}Z_{\sigma 20}\right);\\
A_{\sigma 20} &= (4\pi)^{-1/2}\left(X_{\sigma 00}Z_{\sigma 20} + X_{\sigma 20}Z_{\sigma 00}\right) + \frac{1}{7}\sqrt{\frac{5}{\pi}}X_{\sigma 20}Z_{\sigma 20};\\
G_{\sigma 00} &= \left(\varepsilon\sigma + \frac{1}{6}\varepsilon^2 r^2\right)X_{\sigma 00} - \frac{\varepsilon^2 r^2}{6\sqrt{5}}X_{\sigma 20};\\
G_{\sigma 20} &= \left(\varepsilon\sigma + \frac{5}{42}\varepsilon^2 r^2\right)X_{\sigma 20} - \frac{\varepsilon^2 r^2}{6\sqrt{5}}X_{\sigma 00};\\
F_{\sigma 00} &= (4\pi)^{-1/2}C_{\sigma 20}r^2 X_{\sigma 20};
\end{aligned} \qquad (101)$$

$$F_{\sigma 20} = (4\pi)^{-1/2} C_{\sigma 20} r^2 X_{\sigma 00} + \frac{1}{7}\sqrt{\frac{5}{\pi}} C_{\sigma 20} r^2 X_{\sigma 20}$$

Using the relations (101) and the notation

$$(4\pi)^{-1/2}\left(X_{\sigma l0}; Z_{\sigma l0}; C_{\sigma l0}\right) = \left(X_l; Z_l; C_l\right);\ l = 0,2$$

$$V_0 = \frac{Z_0}{r} + U(0) + \frac{1}{6}\varepsilon^2 r^2 + \varepsilon\sigma;$$

$$V_2 = \frac{6}{r^2} + V_0 + \frac{2\sqrt{5}}{7} U_2;\ U_2 = \frac{Z_2}{r} + \left(C_2 - \frac{\varepsilon^2}{6\sqrt{5}}\right) r^2.$$

one gets the set equations (cf. (68) [9]):

$$\left(\frac{d^2}{dr^2} + E - V_0\right) X_0 = U_2 X_2;$$

$$\frac{d^2}{dr^2} Z_0 = -8\pi\beta N \frac{1}{r}\left(X_0^2 + X_2^2\right);$$

$$\left(\frac{d^2}{dr^2} + E - V_2\right) X_2 = U_2 X_0; \qquad (102)$$

$$\left(\frac{d^2}{dr^2} - \frac{6}{r^2}\right) Z_2 = -16\pi\beta N \frac{1}{r}\left(X_0 X_2 + \frac{\sqrt{5}}{7} X_2^2\right)$$

It is seen from (102) that quantities V_0 and V_2 play the part of the potential energy for solitons corresponding to the main and second harmonics of the wave function. Their asymptotic expressions at $r \to \infty$ are of the form:

$$V_0 = \frac{2\beta N}{r} + \frac{1}{6}\varepsilon^2 r^2 + \varepsilon\sigma;\ U_2 = -\frac{\varepsilon^2}{6\sqrt{5}} r^2;$$

$$V_2 = \frac{6}{r^2} + \frac{2\beta N}{r} + \frac{5}{42}\varepsilon^2 r^2 + \varepsilon\sigma \qquad (103)$$

According to (102) and (103) the expansion of the wave function in perturbation series with respect to magnetic field is valid under the condition

$$\frac{1}{6}\varepsilon^2 R^2 << \left|E_0\right| \qquad (104)$$

If, in addition, the inequality $2/R << \left|E_0\right|$ is fulfilled, then the first and third equations of the set (102) at large r (but less than R) can be written in the form

$$\left(\frac{d^2}{dr^2} + E_0\right) X_0 = 0;\quad \left(\frac{d^2}{dr^2} + E_0\right) X_2 = -\frac{\varepsilon^2 r^2}{6\sqrt{5}} X_0 \qquad (105)$$

The solution of (105) is given by the formula (for $\lambda^{-1} << r << R$)

$$X_0 = a_0 e^{-\lambda r}; \qquad (106)$$

$$X_2 = \left[b_0 - \frac{br}{4\lambda^3}\left(1 + \lambda r + \frac{2}{3}\lambda^2 r^2\right)\right] e^{-\lambda r}.$$

where $\lambda = |E_0|^{1/2}$, $b = -\frac{\varepsilon^2}{6\sqrt{5}} a_0$, a_0 and b_0 are constants. As is seen from (106), the width of the localization region of the second harmonic is somewhat greater than for the main one.

The sought-for solutions of (102) obey the conditions

$$4\pi \int_0^\infty dr\left(X_0^2 + X_2^2\right) = 1,$$

$$C_2 = -\frac{1}{5r^3}\left(2Z_2(r) + r\frac{d}{dr}Z_2(r)\right)\Bigg|_{r\to\infty}$$

and for $r \to 0$ take the form (a and f are constants)

$$X_0 = ar\left(1 + a_1 r^2 + ...\right); \quad Z_0 = -\frac{4\pi}{3}\beta N a^2 r^3 + ...;$$
$$X_2 = r^3\left(f + f_1 r^2 + ...\right); \quad Z_2 = gr^5 + ...$$

where

$$a_1 = -\frac{C}{6}; \quad f_1 = -\frac{1}{14}Cf; \quad g = -\frac{8}{7}\pi\beta Naf;$$
$$C = E - U(0) - \varepsilon\sigma; \quad U(0) = -\frac{d}{dr}Z_0(r)\Bigg|_{r\to\infty}$$

Using the formulae like (100), one calculates the potential energy and the mean square of the orbital angular momentum of the electron

$$W = W_0 + W_2;$$
$$W_0 = N\left[\frac{1}{2}U(0) + 2\pi\int_0^\infty dr\,\frac{1}{r}Z_0(r)\left(X_0^2(r) + X_2^2(r)\right)\right];$$
$$W_2 = 4\pi N\int_0^\infty dr\left(C_2 r^2 + \frac{1}{r}Z_2(r)\right)X_2(r)\left(X_0(r) + \frac{\sqrt{5}}{7}X_2(r)\right);$$
$$L^2 = 24\pi\int_0^\infty dr\,X_2^2(r)$$

The total energy of the electron is given by the formula

$$\mathscr{E}_\sigma = \mathscr{E} + \varepsilon\sigma, \quad \mathscr{E} = C + U(0) + W \quad (\sigma = \pm 1)$$

The results of the numerical calculation of the set of equations (102) are depicted in Fig. 5 (it was assumed that $\beta = \sqrt{2}$, $\varepsilon = 10^{-3}$, $r^* = 10^{-9}$).

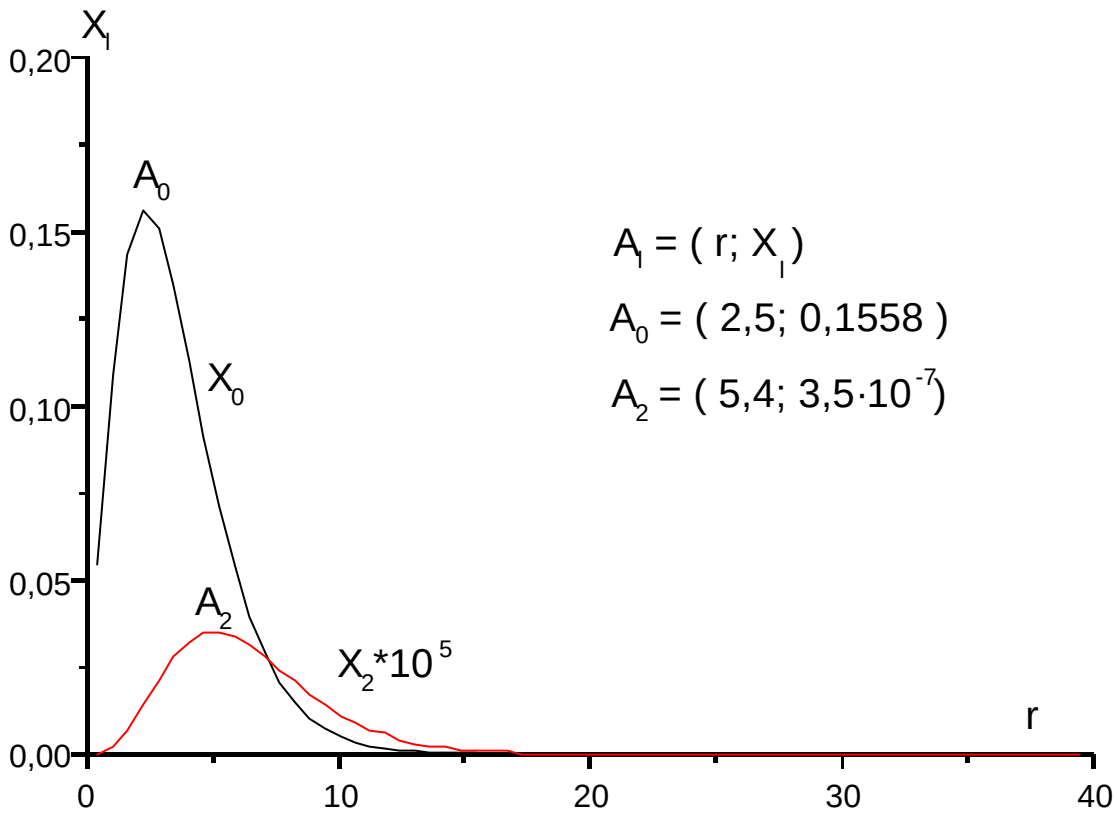

Fig. 5 The components of electron wave function $X_l = X_l(r)$ $(l = 0,2)$ in a magnetic field (the points $A_l = (r; X_l)$ determine the position of maxima)

The quantum state of the electron is characterized by the following magnitudes of the parameters:

a = 0, 111 556 552	$U(0)$ = -1, 262 007
C = 0, 610 928 428	W_0 = 0, 434 055
f = 1, 725 128· 10^{-1}	W_2 = 4,3· 10^{-13}
C_2 = -8, 635· 10^{-8}	$\mathscr{E}$ = -0, 217 024
	L = 0, 67· 10^{-5}

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Appendix

Transformation properties of the self-acting electron field

under the Galilean transformation

Let us consider the inertial frame of reference K' moving relative to the reference frame K with a velocity $\vec{v}_0 = const$. The space and time coordinates in these reference frames are connected with each other by the Galilean transformation

$$\vec{r}' = \vec{r} - \vec{v}_0 t, \quad t' = t \qquad (A.1)$$

We demand that the fundamental equation of motion be form-invariant under the transformation (A.1). This means that if in the reference frame K' the electron wave function $\Psi'(\vec{r}', t')$ obeys the equation

$$i\frac{\partial}{\partial t'}\Psi'(\vec{r}', t') = \left(-\frac{\vec{\nabla}'^2}{2M} + U'(\vec{r}', t')\right)\Psi'(\vec{r}', t'), \qquad (A.2)$$
$$U'(\vec{r}', t) = \beta e^2 \int d\vec{r}_1' \left|\vec{r}' - \vec{r}_1'\right|^{-1} \left(\tilde{\Psi}'^*(\vec{r}_1', t')\Psi'(\vec{r}_1', t') + c.c.\right),$$

then in K the wave function of the particle $\Psi(\vec{r}, t)$ must satisfy the equation

$$i\frac{\partial}{\partial t}\Psi(\vec{r}, t) = \left(-\frac{\vec{\nabla}^2}{2M} + U(\vec{r}, t)\right)\Psi(\vec{r}, t), \qquad (A.3)$$
$$U(\vec{r}, t) = \beta e \int d\vec{r}_1 \left|\vec{r} - \vec{r}_1\right|^{-1} \left(\tilde{\Psi}^*(\vec{r}_1, t)\Psi(\vec{r}_1, t) + c.c.\right)$$

The transformation law for the wave function is written as

$$\Psi(\vec{r}, t) = \Lambda\Psi'(\vec{r}', t') \qquad (A.4)$$

where $\Lambda = \Lambda(\vec{r}', t')$ is the sought-for function obeying the condition

$$\Lambda^+\Lambda = 1 \qquad (A.5)$$

Substituting (A.4) into the second of Eqs. (A.3), one can easily show that

$$U(\vec{r}, t) = U'(\vec{r}', t') \qquad (A.6)$$

We next substitute (A.4) into (A.3) and take into account the equalities (A.1) and (A.6) and the relationships

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \vec{v}_0 \frac{\partial}{\partial \vec{r}'}; \quad \frac{\partial}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}'}$$

As a result, we obtain

$$i\left[\left(\frac{\partial}{\partial t'} - \vec{v}_0 \frac{\partial}{\partial \vec{r}'}\right)\Lambda\right]\Psi' - \left[i\Lambda\vec{v}_0 - \frac{\left(\vec{\nabla}'\Lambda\right)}{M}\right]\vec{\nabla}'\Psi' + \frac{\left(\vec{\nabla}'^2\Lambda\right)}{2M}\Psi' = 0 \qquad (A.7)$$

The requirement that all the terms in (A.7) involving $\vec{\nabla}'\Psi'$ vanish leads to the equation

$$\vec{\nabla}'\Lambda = iM\vec{v}_0\Lambda$$

which may be solved by the substitution

$$\Lambda = \exp\left(iM\,\vec{v}_0\vec{r}'\right) f\left(t'\right) \qquad (A.8)$$

Substituting (A.8) into (A.7) yields the equation

$$i\frac{\partial f}{\partial t'} = -\frac{1}{2}M\,\vec{v}_0^{\,2}\,f$$

with the solution

$$f = C\exp\left(i\frac{M\,\vec{v}_0^{\,2}}{2}t'\right) \qquad (A.9)$$

Constant C is defined by the condition (A.5): $C = 1$ up to an unessential numerical phase factor. Thus

$$\Lambda = \exp\left(i\frac{1}{2}M\,\vec{v}_0^{\,2}t' + iM\,\vec{v}_0\vec{r}'\right) \qquad (A.10)$$

If in the reference frame K' the electron state is described by the wave function

$$\Psi'\left(\vec{r}',t'\right) = e^{-iEt'}\varphi\left(\vec{r}'\right), \quad E = const,$$

then by virtue of (A.4) and (A.10) the electron wave function in K is given by

$$\Psi\left(\vec{r},t\right) = \exp\left\{-i\left(E + \frac{M\,\vec{v}_0^{\,2}}{2}\right)t + iM\,\vec{v}_0\vec{r}\right\}\varphi\left(\vec{r} - \vec{v}_0 t\right) \qquad (A.11)$$

Note that the symmetry being studied of the equation of motion for the electron field is approximate. Indeed, if the electron state in the reference frame K' is stationary, i. e. the particle treated as a unit is at rest, then the electric current and the vortex electric field in this state vanish $\vec{j}' = \vec{E}'_\perp = 0$. However, in view of the Lorents transformation, there will appear in K both the electric current and the vortex electric field $\left(\vec{j} \neq 0,\ \vec{E}_\perp \neq 0\right)$ which were not taken into account in Eq. (A.3). Hence equalities (A.4) and (A.10) along with Eq. (A.3) are valid only provided that the vortex fields $\vec{j}_\perp$ and $\vec{E}_\perp$ are neglected. To estimate the accuracy with which Eq. (A.3) is form-invariant, let us rewrite Eq. (9) in terms of the dimensionless time t' and radius-vector $\vec{r}'$ (see the notation (71)):

$$i\frac{\partial}{\partial t'}\begin{pmatrix}\Psi'\\ \tilde{\Psi}'\end{pmatrix} = \left\{\left(-i\frac{\partial}{\partial\vec{r}'} - \frac{\alpha}{2}\beta\vec{A}'\right)^2 - \frac{\alpha}{2}\beta\vec{\sigma}\left(\frac{\partial}{\partial\vec{r}'}\times\vec{A}'\right) + \beta A'^0\right\}\begin{pmatrix}\Psi'\\ \tilde{\Psi}'\end{pmatrix} \qquad (A.12)$$

Here the quantities Ψ' and A'^μ are defined by (71). According to (A.12) the terms in the equation of motion involving the vortex field are of the order of α^2 (as $\vec{A}' \approx \alpha$). Thus, the symmetry of the equation of motion considered above is valid provided the α^2-order terms are ignored.

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