Interacting Two-Time Physics Field Theory With a BRST Gauge Invariant Action[1]

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Abstract

We construct a field theoretic version of 2T-physics including interactions in an action formalism. The approach is a BRST formulation based on the underlying $\mathrm{Sp}(2,R)$ gauge symmetry, and shares some similarities with the approach used to construct string field theory. In our first case of spinless particles, the interaction is uniquely determined by the BRST gauge symmetry, and it is different than the Chern-Simons type theory used in open string field theory. After constructing a BRST gauge invariant action for 2T-physics field theory with interactions in $d+2$ dimensions, we study its relation to standard 1T-physics field theory in $(d-1)+1$ dimensions by choosing gauges. In one gauge we show that we obtain the Klein-Gordon field theory in $(d-1)+1$ dimensions with unique $\mathrm{SO}(d,2)$ conformal invariant self interactions at the classical field level. This $\mathrm{SO}(d,2)$ is the natural linear Lorentz symmetry of the 2T field theory in $d+2$ dimensions. As indicated in Fig.1, in other gauges we expect to derive a variety of $\mathrm{SO}(d,2)$ invariant 1T-physics field theories as gauge fixed forms of the same 2T field theory, thus obtaining a unification of 1T-dynamics in a field theoretic setting, including interactions. The BRST gauge transformation should play the role of duality transformations among the 1T-physics holographic images of the same parent 2T field theory. The availability of a field theory action opens the way for studying 2T-physics with interactions at the quantum level through the path integral approach.

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1 $\mathrm{Sp}(2, R)$ gauge symmetry and 2T-physics

The essential ingredient in 2T-physics is the basic gauge symmetry $\mathrm{Sp}(2, R)$ acting on phase space X^M, P_M. Under this gauge symmetry, momentum and position are locally indistinguishable, so the symmetry leads to some deep consequences. Some of the phenomena that emerge include certain types of dualities, holography and emergent spacetimes.

The simplest model of 2T-physics is defined by the worldline action [1][2]

$$S_{2T} = \frac{1}{2} \int d\tau \; D_\tau X_i^M X_j^N \eta_{MN} \varepsilon^{ij} = \int d\tau \; \left(\dot{X}^M P^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN}. \tag{1}$$

Here $X_i^M = \left(X^M(\tau), \; P^M(\tau) \right), \; i = 1, 2$, is a doublet under $\mathrm{Sp}(2, R)$ for every M, the structure $D_\tau X_i^M = \partial_\tau X_i^M - A_i{}^j X_j^M$ is the Sp(2,R) gauge covariant derivative, Sp(2,R) indices are raised and lowered with the antisymmetric $\mathrm{Sp}(2, R)$ metric ε^{ij}, and the symmetric $A^{ij}(\tau)$ is the gauge field. In the last expression an irrelevant total derivative $-(1/2) \partial_\tau (X \cdot P)$ is dropped from the action.

The $\mathrm{Sp}(2, R)$ gauge symmetry renders the solutions for $X^M(\tau), P^M(\tau)$ trivial unless there are two timelike dimensions. Therefore the target spacetime metric η^{MN} must have $(d, 2)$ signature. So, the two timelike dimensions is not an input, rather it is an output of the gauge symmetry. $\mathrm{Sp}(2, R)$ is just sufficient amount of gauge symmetry to remove ghosts due to the two timelike dimensions, therefore for a ghost free theory with $\mathrm{Sp}(2, R)$ gauge symmetry one cannot admit more than two timelike dimensions. Although the 2T theory is in $d + 2$ dimensions, there is enough gauge symmetry to compensate for the extra $1 + 1$ dimensions, so that the physical (gauge invariant) degrees of freedom are equivalent to those encountered in 1T-physics in $(d - 1) + 1$ dimensions.

One of the strikingly surprising aspects of 2T-physics is that a given $d + 2$ dimensional 2T theory descends, through $\mathrm{Sp}(2, R)$ gauge fixing, down to a family of holographic 1T images in $(d - 1) + 1$ dimensions, all of which are gauge equivalent to the parent 2T theory and to each other. However, from the point of view of 1T-physics each image appears as a different dynamical system with a different Hamiltonian. Fig.1 below illustrates a family of holographic images that have been obtained from the simplest model of 2T-physics [3]. The central circle represents the 2T action in Eq.(1), while the surrounding ovals represent examples of 1T dynamical systems in $(d - 1) + 1$ dimensions that emerge from the same theory. The 1T systems include interacting as well as free systems in 1T-physics.

Hence 2T-physics can be viewed as a unification approach for one-time physics (1T-physics) systems through higher dimensions. It is distinctly different than Kaluza-Klein theory because there are no Kaluza-Klein towers of states, but instead there is a family of 1T systems with duality type relationships among them.

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[2]The simplest 2T-physics action is in flat $d + 2$ spacetime. More generally 2T-physics is defined in the presence of arbitrary background fields, including electromagnetism, gravity and high spin fields [2].
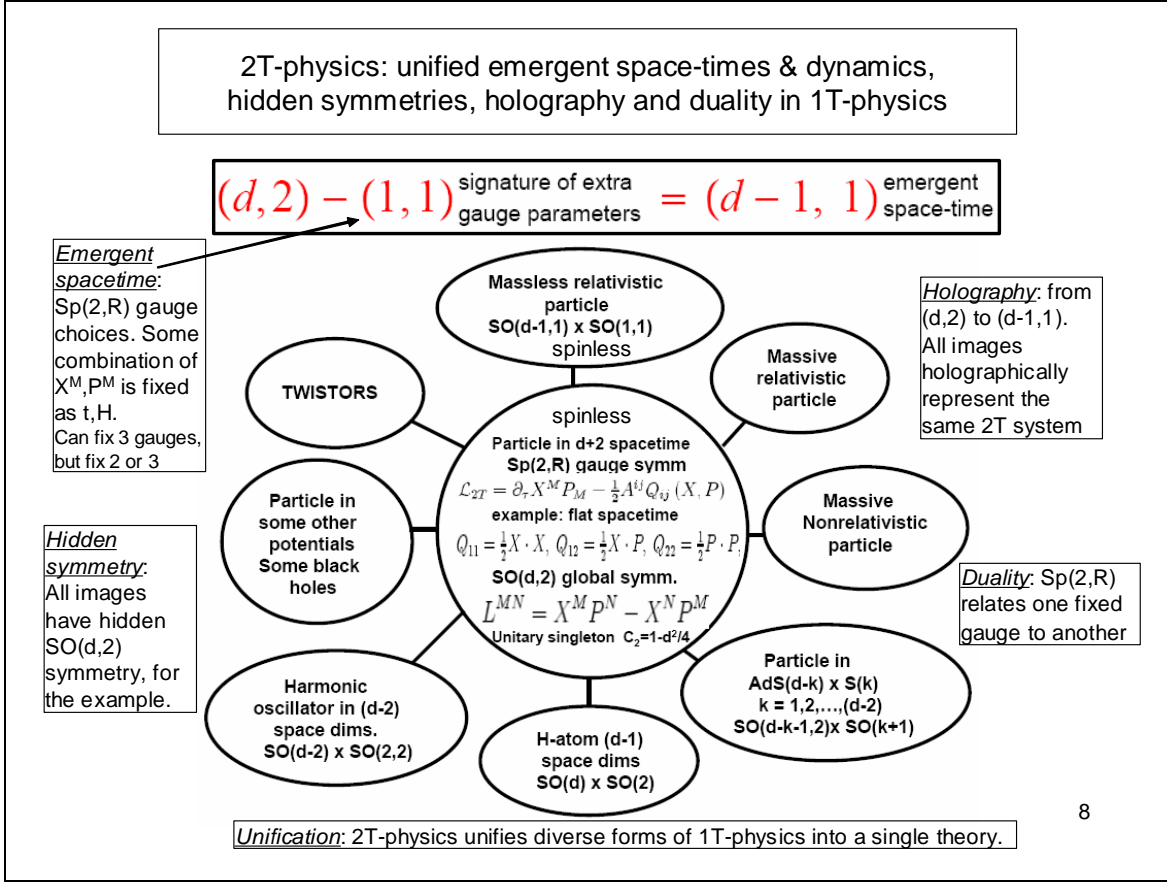
Fig.1 - Some 1T-physics systems that emerge from the solutions of $Q_{ij} = 0$.

In this paper we are interested in constructing a field theoretic formulation of 2T-physics to explore the field theoretic counterpart of the type of phenomena summarized in Fig.1. Previous field theoretic efforts in 2T-physics with interactions are described in [4][5]. Here we will use the results of [4] as a guide for self-consistent equations of motion with interactions, to build an action principle by using a BRST approach similar to the one used to construct string field theory [6]. The BRST action opens the way for discussing the 2T quantum field theory via the path integral.

2 Covariant quantization and field equations

The generators of $\mathrm{Sp}(2,R)$ at the quantum level are $Q_{ij} = \frac{1}{2}X_{(i} \cdot X_{j)}$. Their explicit form in terms of canonical variables $X_i^M = \left(X^M \; P^M\right)$ in $(d+2)$ dimensions is

$$Q_{11} = \frac{1}{2}X^2, \;\; Q_{12} = Q_{21} = \frac{1}{4}\left(X \cdot P + P \cdot X\right), \;\; Q_{22} = \frac{1}{2}P^2. \tag{2}$$

They can be rewritten in terms of the standard notation for $\mathrm{Sp}(2,R) = SL(2,R)$ generators $J_m = (J_0, J_1, J_2)$ as

$$J_0 = \frac{1}{4}\left(P^2 + X^2\right), \;\; J_1 = \frac{1}{4}\left(P^2 - X^2\right), \;\; J_2 = \frac{1}{4}\left(X \cdot P + P \cdot X\right) \tag{3}$$

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Here J_0 is the compact generator. The Lie algebra that follows from the canonical commutation rules $\left[X^M, P^N\right] = i\eta^{MN}$ is

$$[J_m, J_n] = i\varepsilon_{mnk}J^k, \text{ metric } \eta^{mn} = diag\left(-1, 1, 1\right), \ \varepsilon_{012} = +1, \ J^k = \eta^{kl}J_l. \tag{4}$$

The covariant quantization of the theory is defined by the physical states that are annihilated by the $\mathrm{Sp}(2, R)$ generators $J_m|\phi\rangle = 0$. This means that the physical states $|\phi\rangle$ are $\mathrm{Sp}(2, R)$ gauge invariant. In position space the probability amplitude $\phi_0\left(X\right) \equiv \langle X|\phi\rangle$ must satisfy

$$J_m\phi_0\left(X\right) = 0 \text{ or } Q_{ij}\phi_0\left(X\right) = 0. \tag{5}$$

Taking into account that P_M acts like a derivative $P_M\phi_0\left(X\right) = -i\frac{\partial\phi_0(X)}{\partial X^M}$, we obtain the following differential equations that determine the physical states

$$X^2\phi_0\left(X\right) = 0, \ \left(X \cdot \frac{\partial}{\partial X} + \frac{\partial}{\partial X} \cdot X\right)\phi_0\left(X\right) = 0, \ \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}\phi_0\left(X\right) = 0. \tag{6}$$

The first equation has the solution $\phi_0\left(X\right) = \delta\left(X^2\right)A\left(X\right)$, where $A\left(X\right)$ is defined up to additional terms proportional to X^2 (since those vanish when multiplied with $\delta\left(X^2\right)$). In examining the remaining equations we take into account the following properties of the delta function as a distribution (i.e. under integration with smooth functions of X^M in $d + 2$ dimensions)

$$X^2\delta\left(X^2\right) = 0 \tag{7}$$

$$X \cdot \frac{\partial}{\partial X}\delta\left(X^2\right) = 2X^2\delta'\left(X^2\right) = -2\delta\left(X^2\right) \tag{8}$$

$$\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}\delta\left(X^2\right) = 2\left(d + 2\right)\delta'\left(X^2\right) + 4X^2\delta''\left(X^2\right) = 2\left(d - 2\right)\delta'\left(X^2\right) \tag{9}$$

Then, in $d + 2$ dimensions the remaining differential equations in Eq.(6) reduce to the form

$$\delta\left(X^2\right)\left(X \cdot \frac{\partial A}{\partial X} + \frac{d - 2}{2}A\right) = 0, \ \delta\left(X^2\right)\frac{\partial^2 A}{\partial X_M \partial X^M} + 4\delta'\left(X^2\right)\left(X \cdot \frac{\partial A}{\partial X} + \frac{d - 2}{2}A\right) = 0. \tag{10}$$

Therefore $A\left(X\right)$ must satisfy the following differential equations

$$\left(X \cdot \frac{\partial A}{\partial X} + \frac{d - 2}{2}A\right)_{X^2 = 0} = 0, \ \left(\frac{\partial^2 A}{\partial X_M \partial X^M}\right)_{X^2 = 0} = 0. \tag{11}$$

up to additional terms proportional to X^2 . The derivatives in these expressions must be taken before applying $X^2 = 0$.

As indicated above there is freedom in the choice of $A\left(X\right)$, its first derivative $X \cdot \frac{\partial A}{\partial X}$ and its second derivative $\frac{\partial^2 A}{\partial X_M \partial X^M}$ up to arbitrary smooth functions proportional to X^2. This freedom amounts to a gauge symmetry. If this freedom is used so that the extra terms are all gauge fixed to be zero, then the equations above indicate that the gauge fixed $A\left(X\right)$ must be homogeneous of degree $-\frac{d-2}{2}$ and must satisfy the Klein-Gordon equation in $d + 2$ dimensions.

As shown in [4], there are many ways of parameterizing X^M in $(d, 2)$ dimensions that solve $X^2 = 0$. The 1T-physics physical interpretation of the physical states $\phi(X)$ depends on which of the components of X^M is taken to parameterize "time" in 1T-physics. One of these cases was discussed by Dirac [7], and shown to correspond to the massless particle described by the Klein-Gordon equation in $(d-1, 1)$ dimensions. But as shown in [4] there are many other choices of "time" that yield other differential equations which describe a variety of dynamical systems in 1T-physics in $(d-1, 1)$ dimensions. Some of the emerging 1T systems are shown in Fig.1.

As an illustration we show here how the massless Klein-Gordon equation in $(d-1, 1)$ emerges. We use the 2T flat metric $\eta^{MN} = diag(\overset{0'}{-1}, \overset{1'}{+1}, \overset{0}{-1}, \overset{1}{1} \cdots, \overset{d-1}{1})$ in $(d, 2)$ dimensions, and define the lightcone type coordinates $X^{\pm'} = \frac{1}{\sqrt{2}}\left(X^{0'} \pm X^{1'}\right)$ so that $X^M X^N \eta_{MN} = -2X^{+'}X^{-'} + X^\mu X_\mu$. We parametrize X^M in the form

$$X^{+'} = \kappa, \ X^{-'} = \kappa\lambda, \ X^\mu = \kappa x^\mu. \tag{12}$$

The solution to $X^2 = X^M X^N \eta_{MN} = 0$ is then given by

$$\lambda = \frac{x^2}{2}. \tag{13}$$

Then we compute the spacetime metric $ds^2 = dX^M dX^N \eta_{MN}$ in the space of solutions of $X^2 = 0$, and find the emergent spacetime as follows

$$ds^2 = -2dX^{+'}dX^{-'} + dX^\mu dX^\nu \eta_{\mu\nu}, \ \eta_{\mu\nu} = diag\left(-1, 1, \cdots, 1\right), \tag{14}$$

$$= -2\left(d\kappa\right)\left(\frac{x^2}{2}d\kappa + \kappa x^\nu dx_\nu\right) + \left(x^\mu d\kappa + \kappa dx^\mu\right)\left(x^\nu d\kappa + \kappa dx^\nu\right)\eta_{\mu\nu} \tag{15}$$

$$= \kappa^2 dx^\mu dx^\nu \eta_{\mu\nu}. \tag{16}$$

The metric $ds^2 = dX^M dX^N \eta_{MN}$ and the constraint $X^M X_M = 0$ are both invariant under $SO(d, 2)$, so the emergent spacetime metric $ds^2 = \kappa^2 dx^\mu dx^\nu \eta_{\mu\nu}$ must have the same symmetry in a hidden way and non-linearly realized on the remaining coordinates κ, x^μ. Indeed this is the well known non-linear $SO(d, 2)$ conformal transformations of x^μ under which $dx^\mu dx^\nu \eta_{\mu\nu}$ transforms into itself up to an x-dependent conformal factor. Hence κ transforms by the inverse factor so that $ds^2 = \kappa^2 dx^\mu dx^\nu \eta_{\mu\nu}$ remains invariant.

Now we return to the differential equations in Eq.(11) and analyze them in the parametrization of Eq.(12). The homogeneity condition is solved in the form

$$A\left(X^M\right) = \left(X^{+'}\right)^{-(d-2)/2} F\left(\frac{X^{-'}}{X^{+'}}, \frac{X^\mu}{X^{+'}}\right), \text{ assuming } X^{+'} \neq 0, \tag{17}$$

for any function F. Inserting this into the second equation in (11), first taking the derivatives $\partial/\partial X^M$, and then inserting the form of X^M in Eq.(12) gives the Klein-Gordon equation for the field $\varphi(x)$ in x^μ space in $(d-1) + 1$ dimensions[3] (i.e. in the reduced space with one fewer time

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[3]Here we apply the chain rule, $\frac{\partial}{\partial X^M} = \frac{\partial \kappa}{\partial X^M}\frac{\partial}{\partial \kappa} + \frac{\partial \lambda}{\partial X^M}\frac{\partial}{\partial \lambda} + \frac{\partial x^\nu}{\partial X^M}\frac{\partial}{\partial x^\nu}$, that follows from Eq.(12) and then set $\lambda = x^2/2$ as in Eq.(13). Then, $\left(\frac{\partial^2 A}{\partial X^M \partial X_M}\right)_{X^2=0} = \frac{1}{\kappa^2}\frac{\partial^2 \varphi}{\partial x^\mu \partial x_\mu}$ where Eq.(17) has been used. See section (4.1) in [4] for details.

and one fewer space dimensions)

$$A\left(X^M\right)_{X^2=0} = \kappa^{-(d-2)/2}\varphi\left(x\right), \quad \varphi\left(x\right) \equiv F\left(\frac{x^2}{2}, x\right), \quad \partial^\mu\partial_\mu\varphi\left(x\right) = 0. \tag{18}$$

Hence this form of $A\left(X\right)$, which is fully described by the free Klein-Gordon field $\varphi\left(x\right)$ in $(d-1)+1$ dimensions, is a holographic image of the $d+2$ dimensional 2T-physics system described by the three equations in (6).

We have seen that the $d+2$ dimensional field equations in Eq.(6) can be recast as field equations in $(d-1)+1$ dimensions. Depending on the choice of parametrization used to solve the first two equations, the third equation is cast to a second order equation with a particular choice of time (embedded in $d+2$) that controls the dynamical evolution. This remaining equation corresponds to the wave equation of the quantum system described by one of the surrounding ovals in Fig.1 (with some quantum ordering subtleties). For example, instead of the Klein-Gordon equation, the Laplacian on $\text{AdS}_{d-k}\times\text{S}^k$ emerges as discussed in [4] (this reference contains several more examples). So the set of differential equations in Eq.(6) play the role of the central circle in Fig.1, and exhibit the type of phenomena summarized in the figure, namely emergent spacetime, duality, holography, unification, now in a setting of free field theory rather than classical or quantum mechanics. This result [4] is at the level of equations of motion, and without interactions.

Our aim in this paper is to propose an action principle that yields the system of equations in (6) as a result of minimizing the action, and then to generalize the action by including interactions in the 2T field theory formalism. This will provide an environment to explore the properties of 2T-physics, including emergent spacetime, duality, holography, unification, in the presence of interactions, and also provide the proper framework for quantizing the interacting 2T-physics field theory through the path integral approach.

3 BRST operator for $\text{Sp}(2, R)$

The BRST approach is equivalent to the covariant quantization of the system as discussed in the previous section, but it can be used to help develop a natural framework for constructing an action. The following BRST framework is in position space which is different in detail than the BRST framework in phase space previously discussed in [5]. These ideas ar following discussion inspired by the success of the BRST approach in string field theory [6]. We will see that the free field part of the action has the same form as free string field theory, but the interaction will differ from the Chern-Simons form of string field theory, due to the differences in the structure of the constraints J_m that appear in the BRST operator.

To perform the BRST quantization of the theory we introduce the ghosts (c^m, b_m), with $m = 0, 1, 2$ with the canonical structure

$$\{c^m, b_n\} = \delta^m_n, \quad \text{therefore } b_n = \frac{\partial}{\partial c^n} \text{ on functions of } c^m. \tag{19}$$

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The BRST operator that satisfies $Q^2 = 0$ is

$$Q = c^m J_m - \frac{i}{2} \varepsilon^{mnk} c_m c_n b_k. \tag{20}$$

We introduce the field in position space $\Phi\left(X^M, c^m\right)$ including ghosts, and expand it in powers of c^m

$$\Phi(X, c) = \phi_0(X) + c^m \chi_m(X) + (cc)_m \phi^m(X) + (ccc) \chi_0(X) \tag{21}$$

$$(cc)_m \equiv \frac{1}{2} \varepsilon_{mnk} c^n c^k, \quad (ccc) \equiv \frac{1}{3!} \varepsilon_{mnk} c^m c^n c^k \tag{22}$$

Here (ϕ_0, ϕ^m) are bosonic fields and (χ_m, χ_0) are fermionic fields. Similarly, we introduce a gauge parameter in position space $\Lambda\left(X^M, c^m\right)$ that has the expansion

$$\Lambda(X, c) = \lambda_0(X) + c^m \Lambda_m(X) + (cc)_m \lambda^m(X) + (ccc) \Lambda_0(X). \tag{23}$$

Here (λ_0, λ^m) are fermions and (Λ_m, Λ_0) are bosons. We define the BRST transformation $\delta_\Lambda \Phi = Q\Lambda$ and expand it in powers of c

$$\delta_\Lambda \Phi = Q\Lambda = \left(c^m J_m - \frac{i}{2} \varepsilon^{mnk} c_m c_n \frac{\partial}{\partial c^k}\right) \left(\lambda_0 + c^m \Lambda_m + (cc)_m \lambda^m + (ccc) \Lambda_0\right) \tag{24}$$

$$= 0 + c^m (J_m \lambda_0) + (cc)_m \left[\varepsilon^{mnk} J_n \Lambda_k - i\Lambda^m\right] + (ccc) J_m \lambda^m. \tag{25}$$

By comparing coefficients we obtain the BRST gauge transformation in terms of components

$$\delta_\Lambda \phi_0 = 0, \ \delta_\Lambda \chi_m = J_m \lambda_0, \ \delta_\Lambda \phi^m = \varepsilon^{mnk} J_n \Lambda_k - i\Lambda^m, \ \delta_\Lambda \chi_0 = J_m \lambda^m. \tag{26}$$

Note that the first component of Φ is gauge invariant $\delta_\Lambda \phi_0 = 0$, while the last component of the gauge parameter Λ_0 has no effect on any component of Φ.

Powers of Φ are easily computed as

$$\Phi^n = \phi_0^n + c^m \left[\binom{n}{1} \phi_0^{n-1} \chi_m\right] + (cc)_m \left[\binom{n}{1} \phi_0^{n-1} \phi^m + \binom{n}{2} \phi_0^{n-2} (\chi\chi)^m\right] \tag{27}$$

$$+ (ccc) \left[\binom{n}{1} \phi_0^{n-1} \chi_0 + \binom{n}{2} \phi_0^{n-2} (\phi^m \chi_m) + \binom{n}{3} \phi_0^{n-3} (\chi\chi\chi)\right]. \tag{28}$$

The gauge transformation properties of these components follow most easily from Eq.(26), especially since ϕ_0 is invariant. Hence to specify the transformation properties of the components of $\delta_\Lambda \Phi^n$ we only need the following additional equations that follow from Eq.(26)

$$\delta_\Lambda (\phi^m \chi_m) = \left(\varepsilon^{mnk} J_n \Lambda_k - i\Lambda^m\right) \chi_m + \phi^m J_m \lambda_0, \tag{29}$$

$$\delta_\Lambda (\chi\chi)^m = \varepsilon^{mnk} (J_n \lambda_0) \chi_k \tag{30}$$

$$\delta_\Lambda (\chi\chi\chi) = \frac{1}{2} \varepsilon^{mnk} (J_m \lambda_0) \chi_n \chi_k = (J_m \lambda_0) (\chi\chi)^m. \tag{31}$$

3.1 Physical states, BRST cohomology

The physical states are given by the fields that satisfy the BRST cohomology

$$Q\Phi = 0, \text{ solution identified up to shift } \Phi \to \Phi + Q\Lambda \tag{32}$$

We will show that these conditions will follow from our proposed action.

We want to demonstrate that the general solution of $Q\Phi = 0$ has the form $\Phi_{solution} = \phi_0(X) + Q\Lambda(X,c)$, with $J_m\phi_0(X) = 0$. So the physical field Φ contains a non-trivial $\phi_0(X)$, while all other components described by $Q\Lambda(X,c)$ are considered gauge freedom, and can be dropped if so desired. The equation $J_m\phi_0(X) = 0$ is equivalent to covariant quantization, interpreted as the $Sp(2,R)$ gauge invariance condition of the physical field, and its solution was already discussed in the previous section.

To see that $\Phi(X,c) = \phi_0(X) + Q\Lambda(X,c)$, with $J_m\phi_0(X) = 0$, is the general solution of the BRST equation $Q\Phi = 0$ we examine the components of the equation by expanding in powers of c as follows

$$0 = Q\Phi = \left(c^m J_m - \frac{i}{2}\varepsilon^{mnk}c_m c_n \frac{\partial}{\partial c^k}\right)\left(\phi_0 + c^m\chi_m + (cc)_m\phi^m + (ccc)\chi_0\right) \tag{33}$$

$$= 0 + c^m(J_m\phi_0) + (cc)_m\left[\varepsilon^{mnk}J_n\chi_k - i\chi^m\right] + (ccc)J_m\phi^m. \tag{34}$$

Therefore the component fields $(\phi_0, \chi_m, \phi^m, \chi_0)(X)$ must satisfy

$$J_m\phi_0 = 0, \ \varepsilon^{mnk}J_n\chi_k = i\chi^m, \ J_m\phi^m = 0. \tag{35}$$

Note that there is no condition on the last component χ_0. The general solution for $\chi_m, \varphi^m, \chi_0$ is[4]

$$\chi_m = J_m\lambda_0, \ \phi^m = \varepsilon^{mnk}J_n\Lambda_k - i\Lambda^m, \ \chi_0 = \text{any}, \tag{36}$$

for any $(\lambda_0, \Lambda^m, \chi_0)$. We could also write the arbitrary χ_0 in the form $\chi_0 = J_m\lambda^m$ for an arbitrary λ^m. Not all χ_0 can necessarily be written in this form, so in general the BRST cohomology could be richer in content. However, we emphasize that our action below has the additional gauge symmetry $\Phi \to \Phi + (ccc)\xi(X)$ which is equivalent to changing $\chi_0(X)$ by an arbitrary function $\xi(X)$. This indicates that in our problem χ_0 is pure gauge freedom, and therefore it can be chosen as we have specified.

By comparing to Eqs.(25,26), we conclude that the components $\chi_m, \varphi^m, \chi_0$ of the physical field Φ can be written as $Q\Lambda$, so that the general physical field has the form

$$\Phi = \phi_0 + Q\Lambda, \text{ with } J_m\phi_0 = 0. \tag{37}$$

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[4]To arrive at this form first consider the equations (35) as if J_m is a classical vector rather than an operator. Then $J_m\varphi^m = 0$ would be solved by taking any vector φ^m perpendicular to J_m, namely $\varphi^m = \varepsilon^{mnk}J_n\Lambda_k$ for any vector Λ_k. However, since J_m is an operator, $J_m\varphi^m = \varepsilon^{mnk}J_m J_n\Lambda_k$ does not vanish, but gives $iJ^k\Lambda_k$ as a result of the nonzero $Sp(2,R)$ commutation rules. To compensate for this effect the solution is modified to $\varphi^m = \varepsilon^{mnk}J_n\Lambda_k - i\Lambda^m$, and this clearly satisfies $J_m\varphi^m = 0$. Similarly, the solution to $\varepsilon^{mnk}J_n\chi_k = i\chi^m$ is $\chi_m = J_m\lambda_0$. If J_m had been a classical vector it would have given a vanishing result in $\varepsilon^{mnk}J_n\chi_k = \varepsilon^{mnk}J_n J_k\lambda_0$, but since it is an $Sp(2,R)$ operator, the result is $\varepsilon^{mnk}J_n J_k\lambda_0 = iJ^m\lambda_0 = i\chi^m$, thus satisfying the equation.

Hence the BRST cohomology identifies ϕ_0 as the only physical field. This agrees with the covariant quantization of the 2T-physics theory discussed in the previous section and in [4].

We have shown that the quantum version of 2T-physics is correctly reproduced by the BRST cohomology defined by $Q\Phi = 0$, modulo Φ of the form $Q\Lambda$. This combines the differential equations in (6) into a convenient package $Q\Phi = 0$ that can be derived from an action principle as shown below.

4 Action principle

4.1 Free action

We now propose an action from which we derive the physical state condition $Q\Phi(X, c) = 0$ as an equation of motion. We will also be able to redefine Φ by an arbitrary amount $Q\Lambda$ due to a gauge symmetry. The following form of the free action is inspired from previous work on string field theory [6]

$$S(\Phi) = \int \left(d^{d+2}X \right) \left(d^3 c \right) \left[\bar{\Phi} Q \Phi \right]. \tag{38}$$

Here $\bar{\Phi}$ is the hermitian conjugate field. According to the rules of integration for fermions only the (ccc) term in the product $\bar{\Phi} Q \Phi$ contributes to the integral. Therefore the action is obtained in component form by computing the coefficient of (ccc) in $\bar{\Phi} Q \Phi$. This gives

$$S(\Phi) = \int \left(d^{d+2}X \right) \left[\bar{\phi}_0 J_m \phi^m + \bar{\phi}^m J_m \phi_0 + \varepsilon^{mnk} \bar{\chi}_m J_n \chi_k - i \bar{\chi}_m \chi^m \right] \tag{39}$$

The action is independent of χ_0 because of the gauge symmetry under the substitution $\Phi \rightarrow \Phi + (ccc)\xi(X)$ with an arbitry $\xi(X)$. Taking into account the definition of the J_m in terms of (X, P), with $P = -i\partial/\partial X$, we can do integration by parts. For J_0, J_1 integration by parts just moves the derivatives from Φ to $\bar{\Phi}$. However, for J_2 we need to be more careful with signs. Since $J_2\Phi = -i\frac{1}{4}\left[\partial \cdot (X\Phi) + X \cdot \partial\Phi \right]$ involves a first order derivative, its integration by parts introduces a minus sign, but complex conjugation compensates for it and gives the following form

$$S(\Phi) = \int \left(d^{d+2}X \right) \left[\phi^m \overline{(J_m \phi_0)} + \phi_0 \overline{(J_m \phi^m)} + \varepsilon^{mnk} \chi_m \overline{(J_n \chi_k)} + i\chi^m \bar{\chi}_m \right] \tag{40}$$

$$= \int \left(d^{d+2}X \right) \left(d^3 c \right) \left[\Phi \overline{(Q\Phi)} \right] \tag{41}$$

where $\overline{(J_m \phi_0(X))}$ is the complex conjugate after applying the differential operators J_m on ϕ_0, etc. In obtaining this result we have also inserted extra minus signs in changing orders of two fermions such as $-i\bar{\chi}_m \chi^m = +i\chi^m \bar{\chi}_m$. From this result on integration by parts, we see that the latter form of the Lagrangian $\Phi\overline{(Q\Phi)}$ is the hermitian conjugate of the original expression $\bar{\Phi} Q \Phi$. Furthermore we can apply the same method to argue that we can perform integration by parts for any two fields as follows

$$\int \bar{A} Q B = (-1)^{AB+A+B} \int B \overline{(QA)}. \tag{42}$$

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where the sign factor takes into account the Grassmann parity $(-1)^A$ or $(-1)^B$ of the fields A, B.

Using these properties of the integral we can now study the variation of the action

$$\delta S = \int \left(d^{d+2}X\right)\left(d^3c\right)\left[\overline{(\delta\Phi)}Q\Phi + \bar{\Phi}Q\left(\delta\Phi\right)\right] \tag{43}$$

$$= \int \left(d^{d+2}X\right)\left(d^3c\right)\left[\overline{(\delta\Phi)}Q\Phi + \delta\Phi\overline{(Q\Phi)}\right] \tag{44}$$

where in the second term we have used the property of integration by parts. From this we see that the action yields the desired equations of motion for the general variation $\delta\Phi$

$$Q\Phi = 0, \tag{45}$$

and its hermitian conjugate. Furthermore we can also verify that the action has the gauge symmetry under the special transformation $\delta_\Lambda\Phi = Q\Lambda$

$$\delta_\Lambda S = \int \left(d^{d+2}X\right)\left(d^3c\right)\left[\overline{(\delta_\Lambda\Phi)}Q\Phi + \left(\delta_\Lambda\Phi\right)\overline{(Q\Phi)}\right] \tag{46}$$

$$= \int \left(d^{d+2}X\right)\left(d^3c\right)\left[\overline{(Q\Lambda)}Q\Phi + \left(Q\Lambda\right)\overline{(Q\Phi)}\right] \tag{47}$$

$$= \int \left(d^{d+2}X\right)\left(d^3c\right)\left[\overline{\Lambda}Q^2\Phi + \left(Q^2\Lambda\right)\bar{\Phi}\right] = 0. \tag{48}$$

In the third line we have used again the rule for integration by parts and applied the property $Q^2 = 0$. This justifies that the physical state must be taken only up to the gauge transformations. Hence only the ϕ_0 component of Φ contains the physical field since the remainder can be gauge fixed even off-shell. This is consistent with the BRST cohomology of Eq.(32) that we discussed in the previous section, but now we have derived it from an action principle.

It is possible to verify these properties directly in terms of components by using Eqs.(39,40), and we leave this as an exercise for the reader.

4.2   Interactions in equation of motion form

Given the analogy to string field theory, one may be tempted to consider the Cherns-Simons type action by taking a field $(A(X,c))_i{}^j$ that is also a matrix. The expansion in powers of c^m has matrix coefficients $A(X,c) = a_0(X) + c^m\xi_m(X) + (cc)_m a^m(X) + (ccc)\zeta_0(X)$, and the interaction for a Chern-Simons action is

$$S_{cs}(A) = \int \left(d^{d+2}X\right)\left(d^3c\right)\ \mathrm{Tr}\left[\frac{1}{2}AQA + \frac{1}{3}AAA\right]. \tag{49}$$

Unfortunately, this cannot be a valid action for a hermitian $A(X,c)$. The difficulty is rooted in the integration by parts property for J_2 that we discussed following Eq.(39). For a hermitian field the terms involving J_2 drop out after an integration by parts, and therefore the action above cannot be a correct starting point. This is just as well, since we were in the process of discussing

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a scalar field $\Phi(X, c)$ that reduced to the Klein-Gordon field $\varphi(x)$, while the Chern-Simons type action usually is expected to lead to a vector gauge field $(A_\mu(x))_i{}^j$. We expect to return to to the Chern-Simons type action when we discuss spinning particles including a gauge vector field.

So, for the present paper we concentrate on interactions of $\Phi(X, c)$ that will lead to the local self interactions of the scalar field $\varphi(x)$ in $(d-1)+1$ dimensions. The action must still maintain a gauge symmetry of the type $\delta_\Lambda \Phi = Q\Lambda + \cdots$ in the presence of interactions in order to remove the unphysical degrees of freedom ϕ^m, χ_m, χ_0. Here the $(+\cdots)$ is a possible modification of the transformation laws due to the interaction.

We are aiming for the following form of interaction that we know from old work [4] to be consistent at the equation of motion level

$$Q_{11}\phi_0 = \frac{X^2}{2}\phi_0, \ Q_{12}\phi_0 = \frac{X \cdot \partial_X + \partial_X \cdot X}{4i}\phi_0, \tag{50}$$

$$Q_{22}\phi_0 = \frac{1}{2}\left(-\partial_X^2\phi_0 + \Delta\left(\bar{\phi}_0\phi_0\right)\phi_0\right), \text{ where } \Delta\left(\bar{\phi}_0\phi_0\right) = -\tilde{\lambda}\left(\bar{\phi}_0\phi_0\right)^{2/(d-2)}. \tag{51}$$

Only Q_{22} is modified by including the interaction. These interacting Q_{ij} satisfy the $\mathrm{Sp}(2, R)$ commutation rules

$$[Q_{11}, Q_{22}]\phi_0 = 2iQ_{12}\phi_0, \ \ [Q_{12}, Q_{11}]\phi_0 = -iQ_{11}\phi_0, \ \ [Q_{12}, Q_{22}]\phi_0 = iQ_{22}\phi_0, \tag{52}$$

provided ϕ_0 is on shell $Q_{ij}\phi_0 = 0$ (actually only $Q_{11}\phi_0 = Q_{12}\phi_0 = 0$ is enough). Therefore, the following equations of motion can be applied consistently including a unique interaction term, with $\Delta\left(\bar{\phi}_0\phi_0\right)$ given above,

$$Q_{11}\phi_0 = 0, \ \ Q_{12}\phi_0 = 0, \ \ Q_{22}\phi_0 = 0. \tag{53}$$

The uniqueness of the interaction is determined by the consistency conditions of Eq.(52) as follows.

The solution of $Q_{11}\phi_0 = 0$ is still $\phi_0 = \delta(X^2) A(X)$ as before (section 2). Requiring this to satisfy $Q_{12}\phi_0 = 0$ is equivalent to demanding A to have a definite dimension, again same as before (section 2)

$$X \cdot \partial_X A = -\frac{d-2}{2}A. \tag{54}$$

To check the consistency of the interacting case as in Eq.(52) we also need to compute Q_{12} applied on functionals of ϕ_0, such as powers

$$\phi_0^n = \left(\delta\left(X^2\right) A(X)\right)^n = \frac{\delta\left(X^2\right)}{\delta(0)}\left(\delta(0) A(X)\right)^n = \sigma^{n-1}\delta\left(X^2\right)(A(X))^n, \tag{55}$$

where for convenience[5] we have defined $\sigma \equiv \delta(0)$. Such powers ϕ_0^n have anomalous dimensions that shift the dimension away from the naive dimension, namely $X \cdot \partial_X \phi_0^n \neq -n\frac{d-2}{2}\phi_0^n$. This is

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[5]As explained following Eq.(114), after absorbing the factors of σ, the renormalized coupling $\lambda = \tilde{\lambda}\sigma^{4/(d-2)}$ that appears in the interaction terms among the fields $A(X)$ or $\varphi(x)$ is finite, so we do not need to be concerned with the value of σ.

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because the X dependence of the delta functions have disappeared into the constant σ except for one factor. This is discussed in more detail in section (4.4). Having noted this, the action of Q_{12} is computed as follows

$$Q_{12}\left(\frac{\phi_0^n}{\sigma^{n-1}}\right) = \left[\frac{X \cdot \partial_X}{2i}\delta\left(X^2\right)\right]A^n + \delta\left(X^2\right)A^{n-1}\left[n\frac{X \cdot \partial_X}{2i}A\right] + \frac{d+2}{4i}\delta\left(X^2\right)A^n \tag{56}$$

$$= i\delta\left(X^2\right)A^n + \delta\left(X^2\right)A^n\frac{-n\left(d-2\right)}{4i} + \frac{d+2}{4i}\delta\left(X^2\right)A^n \tag{57}$$

$$= i\left(\frac{\phi_0^n}{\sigma^{n-1}}\right)\left(1 + \frac{n\left(d-2\right)}{4} - \frac{d+2}{4}\right) \tag{58}$$

$$= i\frac{\left(n-1\right)\left(d-2\right)}{4}\left(\frac{\phi_0^n}{\sigma^{n-1}}\right). \tag{59}$$

This formula is used to verify that the commutation rule $[Q_{12}, Q_{22}]\phi_0 = iQ_{22}\phi_0$ works only when

$$\Delta = -\tilde{\lambda}\left(\bar{\phi}_0\phi_0\right)^{\frac{2}{d-2}}. \tag{60}$$

We have shown that the equations of motion $Q_{11}\phi_0 = 0$, $Q_{12}\phi_0 = 0$, $Q_{22}\phi_0 = 0$, become

$$\left(X \cdot \frac{\partial A}{\partial X} + \frac{d-2}{2}A\right)_{X^2=0} = 0, \quad \left(-\frac{\partial^2 A}{\partial X_M \partial X^M} - \lambda\left(\bar{A}A\right)^{\frac{2}{d-2}}A\right)_{X^2=0} = 0. \tag{61}$$

The λ here is the renormalized constant[5]. These were the interacting equations obtained before in [4], and now we have written them in the form $Q_{ij}\phi_0 = 0$ for ϕ_0 instead of A. In terms of ϕ_0 they have the BRST form, including interactions, so we may expect to derive them from a BRST type action by generalizing the free BRST action to an interacting one, as will be done below.

4.3 Interacting action and gauge symmetry

The action is obtained by starting with the free BRST action and then modifying Q_{ij} as above. It is convenient to rewrite the field components in the basis Q_{ij} of Eq.(2) instead of the basis of the J_m of Eq.(3). Including only ϕ_0 and ϕ^m $(= \phi_{11}, \phi_{12}, \phi_{22}$, in the new notation), the free action in Eq.(39) is generalized to the following interacting action (suppressing the fermionic components χ^m, χ_0 that decouple)

$$S = \int \left(d^{d+2}X\right)\left\{\left[\bar{\phi}_{22}Q_{11}^0\phi_0 - 2\bar{\phi}_{12}Q_{12}^0\phi_0 + \bar{\phi}_{11}\left(Q_{22}^0 + \Delta\right)\phi_0 + h.c.\right] - U\left(\bar{\phi}_0\phi_0\right)\right\}. \tag{62}$$

We included the superscript "0" to emphasize that Q_{ij}^0 are the free $Sp(2, R)$ generators of Eq.(2,6). There are two sources of interactions. One is

$$\Delta = -\tilde{\lambda}\left(\bar{\phi}_0\phi_0\right)^{2/(d-2)}, \tag{63}$$

which is part of $Q_{22} = Q_{22}^0 + \Delta$ as motivated in the previous section by studying the on-shell equations of motion. The second one, $U\left(\bar{\phi}_0\phi_0\right)$, is like a potential energy term given below.

We can actually attempt to take an arbitrary function $\Delta\left(\bar{\phi}_0\phi_0\right)$ rather than the specific one given above. However, after going through all the calculations, we can show that only the specific form of $\Delta\left(\bar{\phi}_0\phi_0\right)$ above is consistent with all the symmetries generated by the SL(2, R) generators. Therefore we start from the beginning with the form given in Eq.(63) to simplify our presentation.

If the gauge transformations were the free ones of Eq.(26) then any potential term $U\left(\bar{\phi}_0\phi_0\right)$ would be allowed since ϕ_0 would be gauge invariant. However, although ϕ_0 is gauge invariant on shell in the interacting theory (since $Q_{ij}\phi_0 = 0$ with the full Q_{ij}), it is not gauge invariant off shell $Q_{ij}\phi_0 \neq 0$. Then the off-shell gauge symmetry discussed below, together with the on-shell consistency of Eqs.(52), require that the form of U should be restricted, such that Δ and U are related by $\frac{\partial U}{\partial \phi_0} \sim \phi_0\Delta$ up to a proportionality constant that will be determined below consistently. Given Δ above, we must have then

$$U = a\tilde{\lambda}\left(\bar{\phi}_0\phi_0\right)^{d/(d-2)},\tag{64}$$

where a is a constant whose significance will be discussed below.

We now discuss the modified gauge symmetry in the presence of interactions. Since the modified Q_{ij} close when ϕ_0 is on shell as in Eq.(52), we might expect that the action has an *off-shell* gauge symmetry generated by the interacting Q_{ij} with some appropriate modification of the transformation rules given in Eq.(26). Indeed we have found that the interacting action is invariant under modified gauge transformations which are obtained as follows

- Substitute the interacting Q_{ij} instead of the free Q_{ij}^0 in the transformation rules of $\delta_\Lambda\phi_{ij}$.

- Include an additional modification denoted by γ in the last term of $\delta_\Lambda\phi_{12}$ as shown in Eqs.(??,68) below.

Including the modifications just noted, the transformation rules of Eq.(26) for ϕ^m get generalized from the free case $\delta\phi_m = \varepsilon_{mnk}J^n\Lambda^k - i\Lambda_m$ to the interacting case by the inclusion of Δ as follows (with change of notation $\phi_{11} = \phi_0 - \phi_1$, $\phi_{22} = \phi_0 + \phi_1$, $\phi_{12} = \phi_2$)

$$\delta_\Lambda\phi_{11} = Q_{11}^0\Lambda_{12} - Q_{12}^0\Lambda_{11} - i\Lambda_{11},\tag{65}$$

$$\delta_\Lambda\phi_{22} = Q_{12}^0\Lambda_{22} - \left(Q_{22}^0 + \Delta\right)\Lambda_{12} - i\Lambda_{22},\tag{66}$$

$$\delta_\Lambda\phi_{12} = \frac{1}{2}Q_{11}^0\Lambda_{22} - \frac{1}{2}\left(Q_{22}^0 + \Delta\right)\Lambda_{11} - i\left(\Lambda_{12} + \gamma\right)\tag{67}$$

where we insert

$$\gamma = \frac{i\Delta}{d-2}\left(\Lambda_{11} - \frac{\phi_0}{\bar{\phi}_0}\bar{\Lambda}_{11}\right).\tag{68}$$

Let's prove this gauge symmetry of the action. The gauge transformation of the action gives

$$\delta_\Lambda S = \int \left(d^{d+2}X\right)\left[\begin{array}{c} \left(Q_{12}^0\Lambda_{22} - \left(Q_{22}^0 + \Delta\right)\Lambda_{12} - i\Lambda_{22}\right)^* Q_{11}^0\phi_0 \\ -2\left(\frac{1}{2}Q_{11}^0\Lambda_{22} - \frac{1}{2}\left(Q_{22}^0 + \Delta\right)\Lambda_{11} - i\left(\Lambda_{12} + \gamma\right)\right)^* Q_{12}^0\phi_0 \\ + \left(Q_{11}^0\Lambda_{12} - Q_{12}^0\Lambda_{11} - i\Lambda_{11}\right)^* \left(Q_{22}^0 + \Delta\right)\phi_0 + h.c. \end{array}\right]\tag{69}$$

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After integrating by parts and collecting coefficients of the Λ_{ij}, one finds that most terms cancel by using the free commutation rules $[Q_{11}^0, Q_{22}^0] = 2iQ_{12}^0$, $[Q_{12}^0, Q_{11}^0] = -iQ_{11}^0$, $[Q_{12}^0, Q_{22}^0] = iQ_{22}^0$, as well as $\Delta Q_{11}^0 - Q_{11}^0 \Delta = 0$ (since $Q_{11}^0 = \frac{1}{2} X^2$ is not a differential operator). The remaining terms involve Δ and γ as follows

$$\delta_\Lambda S = \int \left(d^{d+2}X\right) \left[\left\{ \begin{array}{c} \bar{\Lambda}_{11} \left[\Delta Q_{12}^0 \phi_0 - Q_{12}^0 \left(\Delta \phi_0\right) + i\Delta \phi_0\right] + h.c. \\ -2i\gamma Q_{12}^0 \phi_0 - \left(2i\gamma Q_{12}^0 \phi_0\right)^* . \end{array} \right\} \right] \tag{70}$$

We are aiming to show that the γ, γ^* terms cancel the Δ terms, so we study the Δ terms in more detail. In this expression Q_{12}^0 is the following derivative operator as applied on any function of X^M in $d+2$ dimensions

$$Q_{12}^0 f(X) = -\frac{i}{4} \left(X \cdot \partial_X + \partial_X \cdot X\right) f(X) \tag{71}$$

$$= -\frac{i}{2} X \cdot \partial_X f(X) - \frac{i}{4} (d+2) f(X). \tag{72}$$

If $\phi_0(X)$ were a smooth function, applying this operator in the first line of Eq.(70), and using the chain rule for derivatives, we would obtain the result

$$\Delta Q_{12}^0 \phi_0 - Q_{12}^0 \left(\Delta \phi_0\right) + i\Delta \phi_0 \tag{73}$$

$$= \frac{i}{2} \left(\phi_0 X \cdot \partial_X \Delta + 2\phi_0 \Delta\right) \tag{74}$$

$$= \frac{i}{2} \frac{2\Delta}{d-2} \left[X \cdot \partial \phi_0 + \frac{\phi_0}{\bar{\phi}_0} X \cdot \partial \bar{\phi}_0 + (d-2) \phi_0\right] \tag{75}$$

where the explicit form of Δ in Eq.(63) is used to compute the derivative $X \cdot \partial_X \Delta$.

However, in computing $Q_{12}^0 \left(\Delta \phi_0\right)$ we must be careful about the fact that $\phi_0(X)$ is very singular. Recall that one of the equations of motion $0 = Q_{11}^0 \phi_0 = \frac{1}{2} X^2 \phi_0$, requires the singular configuration $\phi_0(X) = \delta(X^2) A(X)$. Therefore, functionals such as $\Delta \left(\bar{\phi}_0 \phi_0\right) \phi_0$ that appear in Eq.(70) have to be differentiated carefully by taking into account the singular nature. As we will show in the following subsection, this has the effect of producing an anomalous dimension for the functional $\Delta \left(\bar{\phi}_0 \phi_0\right) \phi_0$, such that, instead of the $(d-2) \phi_0$ in the last term of Eq.(75) we obtain $(d+2) \phi_0$. After including the anomalous dimension (see second line of equation below), the Δ terms in $\delta_\Lambda S$ in Eq.(70) become

$$\bar{\Lambda}_{11} \left[\Delta Q_{12}^0 \phi_0 - Q_{12}^0 \left(\Delta \phi_0\right) + i\Delta \phi_0\right] \tag{76}$$

$$= \frac{i}{2} \frac{2\bar{\Lambda}_{11} \phi_0 \Delta}{d-2} \left[X \cdot \partial \ln \phi_0 + X \cdot \partial \ln \bar{\phi}_0 + (d+2)\right] \tag{77}$$

Two significant features of this expression are: first, it is proportional to $\phi_0 \Delta$ which can be obtained as the derivative of the function $U = a\tilde{\lambda} \left(\bar{\phi}_0 \phi_0\right)^{d/(d-2)}$ up to a proportionality constant,

and second, it vanishes on shell because it is proportional to $Q_{12}^0\phi_0, (Q_{12}^0\phi_0)^*$ as shown below

$$X \cdot \partial \ln \phi_0 + X \cdot \partial \ln \bar{\phi}_0 + (d+2) \tag{78}$$

$$= \frac{2i}{\phi_0} \left(\frac{1}{2i} X \cdot \partial \phi_0 + \frac{1}{4i} (d+2) \phi_0 \right) + \frac{2i}{\bar{\phi}_0} \left(\frac{1}{2i} X \cdot \partial \bar{\phi}_0 + \frac{1}{4i} (d+2) \bar{\phi}_0 \right) \tag{79}$$

$$= 2i \left[\frac{1}{\phi_0} Q_{12}^0 \phi_0 - \left(\frac{1}{\phi_0} Q_{12}^0 \phi_0 \right)^* \right] \tag{80}$$

Including the hermitian conjugate the Δ terms can be rewritten in the following form

$$\bar{\Lambda}_{11} \left[\Delta Q_{12}^0 \phi_0 - Q_{12}^0 (\Delta \phi_0) + i \Delta \phi_0 \right] + h.c. \tag{81}$$

$$= -\frac{2\Delta}{d-2} \left[\left(\bar{\Lambda}_{11} - \frac{\phi_0}{\bar{\phi}_0} \Lambda_{11} \right) Q_{12}^0 \phi_0 \right] + h.c. \tag{82}$$

Therefore we can cancel the Δ terms in $\delta_\Lambda S$ in Eq.(70) by taking γ as given in Eq.(68). With this choice Eq.(70) becomes

$$\delta_\Lambda S = 0,$$

off shell.

We have constructed a gauge invariant interacting 2T field theory in $d+2$ dimensions, with the gauge transformation rules for $(\phi_{11}, \phi_{12}, \phi_{22})$ given by Eqs.(65-67), while γ is explicitly given in (68). The gauge symmetry compensates for one extra space and one extra time dimensions, making this field theory free of ghosts, and closely related to 1T-physics field theory in $(d-1)+1$ dimensions. There are many ways of descending from $d+2$ dimensions to $(d-1)+1$ dimensions as illustrated in Fig.1. Therefore, the 2T-physics field theory we have constructed is expected to lead to a variety of 1T-physics field theories, each being a holographic image of the parent theory, and having duality type relations among themselves. The duality transformation is related to the gauge transformations we discussed in this section.

4.4 Anomalous dimension

Keeping in mind the on-shell singular behavior $\phi_0 = \delta(X^2) A(X)$, we will assume that the off shell ϕ_0 that is relevant in our theory must be similarly singular. So, we will assume that the anomalous dimension of the off-shell expression $\Delta(\bar{\phi}_0 \phi_0) \phi_0$ is the same as the one on-shell, so we compute the anomalous dimension for the on-shell quantity. First note that the powers of the delta function $(\delta(X^2))^n$ collapse to a single overall delta function while the rest are evaluated at $\delta(0)$. We will define the constant $\delta(0) \equiv \sigma$ for brevity. Then we can write

$$\Delta(\bar{\phi}_0 \phi_0) \phi_0 = \delta(X^2) \ A \Delta(\sigma^2 \bar{A} A). \tag{83}$$

For Δ as given in Eq.(63) the constant σ can be absorbed away into a redefinition of the coupling constant $\tilde{\lambda}$, so we do not need to be concerned about the infinite value of σ and simply treat it

formally as a constant in the following arguments. Now we compute the first derivative $X^M \frac{\partial}{\partial X^M}$ of this expression (i.e. dimension operator applied on $\Delta\left(\bar{\phi}_0\phi_0\right)\phi_0$)

$$X \cdot \partial \left(\Delta\left(\bar{\phi}_0\phi_0\right)\phi_0 \right) \tag{84}$$

$$= X \cdot \partial \left[\delta\left(X^2\right) \; A \; \Delta\left(\sigma^2 \bar{A} A\right) \right] \tag{85}$$

$$= A\delta\left(X^2\right) \; X \cdot \partial \Delta\left(\sigma^2 \bar{A} A\right) + \Delta\left(\sigma^2 \bar{A} A\right) \; X \cdot \partial \left(A\delta\left(X^2\right)\right) \tag{86}$$

In the last term we can use $X \cdot \partial \delta\left(X^2\right) = -2\delta\left(X^2\right)$ as in Eq.(8) to argue that every term in this expression is proportional to $\delta\left(X^2\right)$. Note that in evaluating $X \cdot \partial \Delta\left(\sigma^2 \bar{A} A\right)$ in what follows, $\sigma = \delta\left(0\right)$ will not contribute a similar factor to $X \cdot \partial \delta\left(X^2\right) = -2\delta\left(X^2\right)$, and this is the subtlety that leads to an anomalous dimension. With this understanding we proceed with the following computation by using the explicit form of Δ given in Eq.(63)

$$\delta\left(X^2\right) \; A \; X \cdot \partial \Delta\left(\sigma^2 \bar{A} A\right) \tag{87}$$

$$= \frac{2}{d-2}\delta\left(X^2\right) \; A \; \Delta\left(\sigma^2 \bar{A} A\right) \; X \cdot \partial \ln\left(\bar{A} A\right) \tag{88}$$

$$= \frac{2}{d-2}\delta\left(X^2\right) \; A \; \Delta\left(\sigma^2 \bar{A} A\right) \; \left[\frac{1}{A}X \cdot \partial A + \frac{1}{\bar{A}}X \cdot \partial \bar{A}\right] \tag{89}$$

$$= \frac{2}{d-2}\Delta\left(\sigma^2 \bar{A} A\right) \; \left[\delta\left(X^2\right) \; X \cdot \partial A + \frac{A}{\bar{A}} \delta\left(X^2\right) \; X \cdot \partial \bar{A}\right] \tag{90}$$

$$= \frac{2}{d-2}\Delta\left(\sigma^2 \bar{A} A\right) \; \left[X \cdot \partial \left(A\delta\left(X^2\right)\right) + \frac{A}{\bar{A}}X \cdot \partial \left(\bar{A}\delta\left(X^2\right)\right) + 4A\delta\left(X^2\right)\right] \tag{91}$$

In the last line we pulled $\delta\left(X^2\right)$ inside the derivative by using again $X \cdot \partial \delta\left(X^2\right) = -2\delta\left(X^2\right)$ as in Eq.(8). Now we combine this result with Eq.(86) and obtain

$$X \cdot \partial \left[A\delta\left(X^2\right) \; \Delta\left(\sigma^2 \bar{A} A\right)\right] \tag{92}$$

$$= \frac{2}{d-2}\Delta\left(\sigma^2 \bar{A} A\right) \left[X \cdot \partial \left(A\delta\left(X^2\right)\right) + 4A\delta\left(X^2\right)\right] + \Delta\left(\sigma^2 \bar{A} A\right) \; X \cdot \partial \left(A\delta\left(X^2\right)\right) \tag{93}$$

$$+ \frac{2}{d-2}\frac{A}{\bar{A}}\Delta\left(\sigma^2 \bar{A} A\right) X \cdot \partial \left(\bar{A}\delta\left(X^2\right)\right) \tag{94}$$

$$= \frac{2}{d-2}\Delta\left(\bar{\phi}_0\phi_0\right) \left[X \cdot \partial\phi_0 + \frac{\phi_0}{\bar{\phi}_0}X \cdot \partial\bar{\phi}_0 + 4\phi_0\right] + \Delta\left(\bar{\phi}_0\phi_0\right) X \cdot \partial\phi_0 \tag{95}$$

In the final line we replaced back $\phi_0 = A\left(X\right)\delta\left(X^2\right)$. In this way we have derived

$$X \cdot \partial \left(\Delta\left(\bar{\phi}_0\phi_0\right)\phi_0\right) = \frac{2}{d-2}\Delta\left(\bar{\phi}_0\phi_0\right) \left[X \cdot \partial\phi_0 + \frac{\phi_0}{\bar{\phi}_0}X \cdot \partial\bar{\phi}_0 + 4\phi_0\right] \tag{96}$$

$$+ \Delta\left(\bar{\phi}_0\phi_0\right) X \cdot \partial\phi_0 \tag{97}$$

The term $4\phi_0$ inside the bracket has no derivatives, so this is the anomalous term that would not arise if we had smooth functions. When this $4\phi_0$ is combined with the $\left(d-2\right)\phi_0$ in Eq.(75) it leads to $\left(d+2\right)\phi_0$ that appears in Eq.(77).

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We computed the anomalous dimension for the on-shell quantity $\Delta\left(\bar{\phi}_0\phi_0\right)\phi_0$ by imposing only one of the on-shell conditions, while the other two conditions are still off-shell. At this stage we do not have a corresponding proof for the fully off-shell quantity, but assumed that the same result holds. In any case it is evident that for field configurations that are singular like $\delta\left(X^2\right)$ the result holds, and for those field configurations the gauge invariance of the action is valid.

5   Standard 1T-field theory from 2T-field theory

Having justified the gauge symmetry, we can now make various gauge choices for ϕ_{ij}. We would like to show that in a specific gauge the 2T-physics field theory in $d+2$ dimensions descends to the standard interacting Klein-Gordon field theory in $(d-1)+1$ dimensions. We emphasize that, in other gauges as in Fig.1, we expect to find other field theories instead of the Klein-Gordon theory, so the discussion below is an illustration of the reduction from 2T-physics to 1T-physics in the context of field theory. This reduction is done in this section at the level of the action and should be compared to the corresponding reduction at the level of equations of motion discussed earlier in this paper and in [4].

The 2T action in Eq.(62) contains the fields ϕ_{ij} and ϕ_0. We must require that the equations of motion that follow from the original action, after gauge fixing, agree with the equations of motion produced by the gauge fixed action. So we start by writing down all equations of motion and then fix the gauges. The variation of the action with respect to $\delta\bar{\phi}_{11}$ before choosing a gauge gives the equation

$$Q_{22}^0\phi_0 + \phi_0\Delta = 0, \text{ with } \Delta = -\tilde{\lambda}\left(\bar{\phi}_0\phi_0\right)^{2/(d-2)}. \tag{98}$$

The variation $\delta\bar{\phi}_0$ before choosing a gauge gives the equation[6]

$$Q_{11}^0\phi_{22} - 2Q_{12}^0\phi_{12} = -Q_{22}^0\phi_{11} - \left[\left(\frac{\phi_{11}d}{\phi_0} + \frac{2\bar{\phi}_{11}}{\bar{\phi}_0}\right) + ad\right]\frac{\phi_0\Delta}{d-2}. \tag{99}$$

where we have used $\frac{\partial U}{\partial\phi_0} = \frac{ad\tilde{\lambda}}{d-2}\phi_0\left(\bar{\phi}_0\phi_0\right)^{2/(d-2)} = -\frac{ad}{d-2}\phi_0\Delta$.

Now we choose the gauge

$$\phi_{11} = \gamma\phi_0, \tag{100}$$

where γ is a constant real coefficient to be determined below through consistency. In this gauge, after taking Eq.(98) into account, Eq.(99) reduces to

$$Q_{11}^0\phi_{22} - 2Q_{12}^0\phi_{12} = -\frac{4\gamma + ad}{d-2}\phi_0\Delta. \tag{101}$$

Hence, demanding that both the $\delta\bar{\phi}_0$ and the $\delta\bar{\phi}_{11}$ variations be compatible in the gauge $\phi_{11} = \gamma\phi_0$ requires that the remaining degrees of freedom ϕ_{12}, ϕ_{22} satisfy the equation above. In addition,

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[6]Observe that, including the hermitian conjugate, the interaction with ϕ_{11} is of the form $-\tilde{\lambda}\bar{\phi}_{11}\left(\bar{\phi}_0\right)^{\frac{2}{d-2}}\left(\phi_0\right)^{\frac{d}{d-2}} - \tilde{\lambda}\left(\bar{\phi}_0\right)^{\frac{d}{d-2}}\left(\phi_0\right)^{\frac{2}{d-2}}\phi_{11}$.

the variation of the original action with respect to ϕ_{12}, ϕ_{22} demand the equations of motion

$$Q_{11}^0 \phi_0 = 0, \quad Q_{12}^0 \phi_0 = 0. \tag{102}$$

If we work on mass shell for these two equations, we can choose gauges for the corresponding fields ϕ_{12}, ϕ_{22}, and those gauge choices must be consistent with Eq.(101). In this way $\phi_{11}, \phi_{12}, \phi_{22}$ get all determined in terms of ϕ_0.

We now return to ϕ_0, which satisfies the equations $Q_{11}^0 \phi_0 = Q_{12}^0 \phi_0 = Q_{22}^0 \phi_0 + \phi_0 \Delta = 0$. We examine how the dynamics of ϕ_0 can be derived consistently from the gauge fixed action. After satisfying the ϕ_{12}, ϕ_{22} equations (namely $Q_{11}^0 \phi_0 = Q_{12}^0 \phi_0 = 0$) as above, and inserting the gauge $\phi_{11} = \gamma \phi_0$, the gauge fixed action in Eq.(62) becomes a functional of only ϕ_0. So the gauge fixed action is

$$S(\phi_0) = \gamma \int \left(d^{d+2} X \right) \left[2 \bar{\phi}_0 Q_{22}^0 \phi_0 - \left(2 + \frac{a}{\gamma} \right) \tilde{\lambda} \left(\bar{\phi}_0 \phi_0 \right)^{\frac{d}{d-2}} \right]. \tag{103}$$

The variation of this action for $\bar{\delta} \phi_0$ gives the equation of motion

$$Q_{22}^0 \phi_0 + \left(1 + \frac{a}{2\gamma} \right) \frac{d}{d-2} \phi_0 \Delta = 0. \tag{104}$$

For consistency this must agree with Eq.(98). Therefore, we must require that the parameter γ that appeared as part of the gauge fixing in Eq.(100) must be determined consistently so that $\left(1 + \frac{a}{2\gamma} \right) \frac{d}{d-2} = 1$, hence

$$\gamma = -\frac{1}{4} da. \tag{105}$$

Thus, the gauge fixed action takes the form

$$S(\phi_0) = -\frac{ad}{2} \int \left(d^{d+2} X \right) \left[\bar{\phi}_0 Q_{22}^0 \phi_0 - \left(1 - \frac{2}{d} \right) \tilde{\lambda} \left(\bar{\phi}_0 \phi_0 \right)^{\frac{d}{d-2}} \right]. \tag{106}$$

In addition to the equation of motion $Q_{22}^0 \phi_0 + \phi_0 \Delta = 0$ that follows from this gauge fixed action, we must also impose the conditions $Q_{11}^0 \phi_0 = Q_{12}^0 \phi_0 = 0$. So, let us determine the action after these kinematic equations have been imposed. In section (2) we have already shown that the solution to $Q_{11}^0 \phi_0 = Q_{12}^0 \phi_0 = 0$ is

$$\phi_0(X) = \delta \left(X^2 \right) \left(X^{+'} \right)^{-(d-2)/2} F \left(\frac{X^{-'}}{X^{+'}}, \frac{X^\mu}{X^{+'}} \right) \tag{107}$$

and that $Q_{22}^0 \phi_0$ takes the form

$$Q_{22}^0 \phi_0 = -\frac{1}{2} \delta \left(X^2 \right) \frac{\partial^2}{\partial X^M \partial X_M} \left[\left(X^{+'} \right)^{-(d-2)/2} F \left(\frac{X^{-'}}{X^{+'}}, \frac{X^\mu}{X^{+'}} \right) \right]. \tag{108}$$

After the change of variables $X^M \to (\kappa, \lambda, x^\mu)$ in Eq.(12) this simplifies as in Eq.(18)

$$Q_{22}^0 \phi_0 = \delta \left(\left(\lambda - \frac{x^2}{2} \right) \kappa^2 \right) \kappa^{-d+2} \frac{1}{2\kappa^2} \left(-\frac{\partial^2 \varphi(x)}{\partial x^\mu \partial x_\mu} \right), \tag{109}$$

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where x^μ are the coordinates in $(d-1)+1$ dimensions and (λ, κ) are the extra coordinates. In these coordinates the first term in the Lagrangian takes the form

$$\bar{\phi}_0 Q_{22}^0 \phi_0 = \sigma \delta \left(\lambda - \frac{x^2}{2} \right) \kappa^{-(d+2)} \left(-\frac{1}{2} \bar{\varphi}(x) \frac{\partial^2 \varphi(x)}{\partial x^\mu \partial x_\mu} \right) \tag{110}$$

where $\sigma = \delta(0)$ as defined before. The interaction term has a similar overall factor that is computed as follows

$$\tilde{\lambda} \left(\bar{\phi}_0 \phi_0 \right)^{\frac{d}{d-2}} = \tilde{\lambda} \left(\delta \left(X^2 \right) \kappa^{-(d-2)/2} \right)^{\frac{2d}{d-2}} (\bar{\varphi}\varphi)^{\frac{d}{d-2}} \tag{111}$$

$$= \tilde{\lambda} \sigma^{\frac{4}{d-2}+1} \delta \left(X^2 \right) \kappa^{-d} (\bar{\varphi}\varphi)^{\frac{d}{d-2}} \tag{112}$$

$$= \sigma \delta \left(\lambda - \frac{x^2}{2} \right) \kappa^{-(d+2)} \lambda (\bar{\varphi}\varphi)^{\frac{d}{d-2}}, \tag{113}$$

where λ is the finite renormalized constant

$$\lambda = \tilde{\lambda} \sigma^{\frac{4}{d-2}}. \tag{114}$$

Inserting these results into the gauge fixed action in Eq.(106), and using

$$\left(d^{d+2} X \right) = \kappa^{d+1} d\kappa d\lambda \left(d^d x \right) \tag{115}$$

we obtain the Klein-Gordon theory as follows

$$S(\varphi) = \left[-\frac{ad}{4} \sigma \left(\int d\kappa \right) \right] \int d\lambda \delta \left(\lambda - \frac{x^2}{2} \right) \int \left(d^d x \right) L(\varphi) \tag{116}$$

where

$$L(\varphi) = \int \left(d^d x \right) \left[-\bar{\varphi} \frac{\partial^2 \varphi}{\partial x^\mu \partial x_\mu} - 2 \left(1 - \frac{2}{d} \right) \lambda (\bar{\varphi}\varphi)^{\frac{d}{d-2}} \right]. \tag{117}$$

For the correct normalization of the Klein-Gordon field $\varphi(x)$ we must renormalize the overall action by the inverse of the overall factor $\left[-\frac{ad}{4} \sigma \left(\int d\kappa \right) \right]$, or else choose the overall extra parameter a that appeared in $U \left(\bar{\phi}_0 \phi_0 \right)$ to tune this overall factor to exactly 1, so that

$$S(\varphi) = \int \left(d^d x \right) L(\varphi). \tag{118}$$

$L(\varphi)$ is the usual conformally invariant *interacting* theory with $\mathrm{SO}(d,2)$ symmetry at the classical level. The interacting equations of motion derived from this 1T action are in agreement with those derived from the original 2T action.

This section developed the methods for gauge fixing the interacting 2T field theory, and one can now apply similar methods to study other gauges, such as the ones indicated in Fig.1.

6 Comments

We have successfully constructed a field theoretic formulation of 2T-physics including interactions. The approach is a $\mathrm{Sp}(2,R)$ BRST formulation [8] based on the underlying $\mathrm{Sp}(2,R)$ gauge symmetry of 2T-physics. Some steps were similar to string field theory [6] and this was used mainly for inspiration. However, the form of the interaction and the methods used are new as a BRST gauge theory formulation. In this we were guided by the previous work at the equation of motion level [4]. We found that the interaction at the level of the action is uniquely determined by the interacting BRST gauge symmetry which is different than the free BRST gauge transformation.

We have shown that with a particular gauge choice the 2T field theory comes down to the standard interacting Klein-Gordon form. As expected from previous work in 2T-physics, we have related the global $\mathrm{SO}(d,2)$ conformal symmetry of the 1T Klein-Gordon theory directly to the global space-time symmetry of the 2T-physics field theory in $d+2$ dimensions, including interactions. Note that the Klein-Gordon theory is just one of the possible holographic images of the 2T field theory. We should be able to derive other holographic images, such as those that appear in Fig.1. The various interacting field theories must then be dual to each other. Through such dualities we may be able to obtain non-perturbative information about various field theories.

The BRST operator we started with included only the constraints Q_{ij}^0 of the worldline action in Eq.(1) and include also the corresponding ghosts c^{ij}, b_{ij} (equivalently c^m, b_m). It is possible to enlarge our BRST operator by also including the constraints due to the fact that the gauge fields A^{ij} have vanishing canonical conjugates $\pi_{ij} = 0$, and the corresponding ghosts $\tilde{c}^{ij}, \tilde{b}_{ij}$. If we had taken that approach[7], then the field Φ would have to be taken as a function of more variables $\Phi\left(X, c^{ij}, A_{ij}, \tilde{c}^{ij}\right)$. This would make our analysis more complete but much more complex, and possibly with no benefit. However, it may still be useful to pursue this more complete approach. Similarly, after getting motivated by the transformation properties of all the fields in the free theory given by Φ of Eq.(21), we decided to concentrate only on ϕ_0 and ϕ_{ij} in constructing the interacting theory, assuming the others decouple. Indeed we found a consistent gauge theory with the smaller set of fields ϕ_0 and ϕ_{ij}. It may be useful to extend our methods to include all the fields in Φ and then rewrite the interacting theory in terms of the package $\Phi\left(X, c^{ij}\right)$. Since this does not seem to be urgent we have left it to future work.

The interacting action that we have suggested provides the first step for studying 2T-physics at the quantum level through the path integral. This may be done without reference to 1T-physics, which would be interesting in its own right, but for interpretation of the results it would be desirable to also consider gauge fixing to 1T-physics.

The steps that we have performed in the gauge fixing process were done at the classical field theory level. It is evident that we can study the corresponding gauge choices in the path integral to relate 2T physics to many versions of 1T physics, including interactions at the quantum field theory level. This remains to be investigated in detail, but we expect to be able to establish

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[7]For a study of this type, but for the much simpler case of the 1T worldline spinless particle, see [9].

physically similar results to the classical version, modulo some possible quantum corrections in various gauges.

The methods used in this paper can be generalized to spinning particles, including gauge fields, to find the general interacting 2T field theory. For this one can use as a guide the previous work at the equation of motion level for interacting spinning particles up to spin 2 [4]. The Chern-Simons version of interactions may be relevant in the case of gauge fields.

Our motivation in developing the current formalism was to work our way in a similar fashion toward building a consistent interacting 2T field theory and to apply the results to construct the Standard Model of Elementary Particles and Forces from the point of view of 2T-physics in a BRST field theory formalism. In fact, the BRST gauged fixed version in Eq.(106) served as a spring board for developing a closely related but simpler formulation of 2T quantum field theory that applies to spins 0,1/2,1,2. This provided a short cut for successfully constructing the Standard Model in the framework of 2T-physics, thus establishing that 2T-physics provides the correct description of Nature from the point of view of 4+2 dimensions [10]. The BRST approach may still provide guidance in the long run to settle remaining issues, as well as to go beyond the Standard Model and toward M-theory.

Acknowledgments

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